

# A new class of exact solutions in string theory

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## Abstract

We prove that a large class of leading order string solutions which generalize both the plane-wave and fundamental string backgrounds are, in fact, exact solutions to all orders in  $\alpha'$ . These include, in particular, the traveling waves along the fundamental string. The key features of these solutions are a null symmetry and a chiral coupling of the string to the background. Using dimensional reduction, one finds that the extremal electric dilatonic black holes and their recently discovered generalizations with NUT charge and rotation are also exact solutions. We show that our bosonic solutions are also exact solutions of the heterotic string theory with no extra gauge field background.

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## 1. Introduction

To address strong field effects in string theory, it is necessary to obtain exact classical solutions and study their properties. As in other field theories, symmetries have been used to help find these solutions. It is easy to show that every Killing vector on spacetime gives rise to a conserved current on the string world sheet. If the antisymmetric tensor field is related to the spacetime metric in a certain way, these currents are chiral. The existence of such chiral currents turns out to simplify the search for exact solutions. One example is the WZW model which describes string propagation on a group manifold. This background has a large symmetry group, and all the associated currents are chiral. (Since the gauged WZW models can be represented in terms of the difference between two WZW models for a group and a subgroup, a similar statement applies there.) Another example is provided by the  $F$ -models discussed in [1,2] which have two null Killing vectors and two associated chiral currents. In addition to these two examples, the only other known exact solutions to (bosonic) string theory are the plane waves and their generalizations [3,4], which are characterized by the existence of a covariantly constant null Killing vector.

We will show that the  $F$ -models and generalized plane waves are both special cases of a larger class of exact solutions which have a null Killing vector and an associated conserved chiral current. Backgrounds of this type are described by  $\sigma$ -models which we will refer to as “chiral null models”. We will see that they include a number of interesting examples.

The presence of a null chiral current is associated with an infinite-dimensional affine symmetry of the  $\sigma$ -model action. This implies special properties of the spacetime fields. The generalized connection with torsion equal to the antisymmetric field strength plays an important role since it is the one that appears in the classical string equations of motion. We will see that this connection has reduced holonomy. A certain balance between the metric and the antisymmetric tensor resulting in chirality of the action is the crucial property of our models which is in the core of their exact conformal invariance.

There are several levels of describing solutions to string theory. The string equation is usually expressed in terms of a power series in  $\alpha'$ . If one keeps only the leading order terms, one obtains an equation analogous to Einstein's equation and a large number of solutions have been found. The form of the higher order terms is somewhat ambiguous due to the freedom of choosing different renormalization schemes (or field redefinitions). For the plane - wave type solutions and the  $F$ -models, it has been shown that there exists a scheme in which the leading order solution does not receive  $\alpha'$  corrections, and thus corresponds to an exact solution as well. We will see that the same is true for the more general chiral null models.

To explore the properties of a given solution, one would like to know not only that a given background is an exact solution to the field equations, but also what the string states and interactions are in this background. In other words, one would like to know the corresponding conformal field theory explicitly. This is known only for gauged WZW models. But some chiral null models can be realized as gauged WZW models [5,1] so in these cases, one has more information about the solution.

Many of the chiral null model backgrounds have unbroken space-time supersymmetry and some models admit extended world sheet supersymmetry. For example, the  $F$ -models in even dimensions always have at least (2,0) world sheet supersymmetry. However, our argument that they are exact string solutions is not based on this fact. We will show that these backgrounds are solutions in the bosonic as well as the superstring and heterotic string theories. What types of solutions belong to this class? To begin, all of the plane

wave type solutions are included, as well as all of the  $F$ -models [1] which contain the fundamental string solution [6] as a special case. In addition, several generalizations of these solutions are in this class, including the traveling waves along the fundamental string [7]. Although the bosonic string does not have fundamental gauge fields, effective gauge fields can arise from dimensional reduction. In this way, we will show that the charged fundamental string solutions [8,9] are exact.

Perhaps of most importance is the fact that four dimensional extremal electrically charged black holes [10,11,12] can be obtained from the dimensional reduction of a chiral null model, and hence are exact. Similarly, we will see that the generalizations of the extremal black holes which include NUT charge and rotation [13,14,15] are also exact. Finally, the chiral null models also describe some backgrounds with magnetic (and no electric) fields, as well as other solutions which appear to be new.

If one considers only the leading order string equations, many of these solutions arise as the extremal limit of a family of solutions with a regular event horizon. The non-extremal solutions are not of the chiral null form and are likely to receive  $\alpha'$  corrections in all renormalization schemes. Finding the exact analogs of these solutions (which include the Schwarzschild metric as a special case) remains an outstanding open problem. The fact that we only obtain a particular charge to mass ratio from a chiral null model can be understood roughly as follows. To have chiral currents, one needs a balance between the spacetime metric and antisymmetric tensor field, which upon dimensional reduction results in a relation between the charge and the mass.

This paper is organized as follows. In the next section we introduce the chiral null models, and discuss their properties as well as some special cases and examples of solutions. In Section 3 we describe a general scheme of Kaluza-Klein type dimensional reduction working directly at the level of the string world sheet action. Unlike the more traditional approach which uses the leading order terms of the spacetime effective action, our approach applies to all orders in  $\alpha'$ . Section 4 will be devoted to solutions obtained from the dimensional reduction of a chiral null model. These include the charged fundamental string, extremal electric black holes and their generalizations.

Section 5 contains our main result: we prove that for a chiral null model, the leading order solutions do not receive any  $\alpha'$  corrections (in a particular scheme). In Section 6 we extend this argument to the case of superstring and heterotic string theory. We show that the  $(1,0)$  supersymmetric extensions of our bosonic models are conformally invariant without any extra gauge-field background. We also discuss the world sheet supersymmetry properties of these models. Section 7 is devoted to some concluding remarks.

In Appendix A we summarize the geometrical properties of the string backgrounds described by the chiral null model (the generalized connection with torsion, its holonomy and curvature tensor, parallelizable spaces, etc.). In Appendix B we elaborate on the discussion of  $D = 3$  models in [1] and show that the general chiral null model in three dimensions is actually a gauged WZW model.

## 2. Chiral null models: general properties and examples

### 2.1. Review of previous work

A bosonic string in a general ‘massless’ background is described (in the conformal gauge) by the  $\sigma$ -model

$$I = \frac{1}{\pi\alpha'} \int d^2z L, \quad L = (G_{MN} + B_{MN})(X) \partial X^M \bar{\partial} X^N + \alpha' \mathcal{R}\phi(X), \quad (2.1)$$

where  $G_{MN}$  is the metric,  $B_{MN}$  is the antisymmetric tensor and  $\phi$  is the dilaton [16] ( $\mathcal{R}$  is related to the world sheet metric  $\gamma$  and its scalar curvature by  $\mathcal{R} \equiv \frac{1}{4}\sqrt{\gamma}R^{(2)}$ ;  $\partial$  and  $\bar{\partial}$  stand for  $\partial_+$  and  $\partial_-$  when the world sheet signature is Minkowskian).

In [1], two types of models were studied, which were called the  $K$ -model and the  $F$ -model. In terms of the coordinates  $X^M = (u, v, x^i)$ , the simplest (flat transverse  $x^i$ -space)  $K$ - and  $F$ -model Lagrangians are

$$L_K = \partial u \bar{\partial} v + K(x) \partial u \bar{\partial} u + \partial x_i \bar{\partial} x^i + \alpha' \mathcal{R} \phi_0, \quad \phi_0 = \text{const}, \quad (2.2)$$

$$L_F = F(x) \partial u \bar{\partial} v + \partial x_i \bar{\partial} x^i + \alpha' \mathcal{R} \phi(x). \quad (2.3)$$

These two models are dual in the sense that applying a spacetime duality transformation [17] with respect to  $u$  turns the  $K$ -model into the  $F$ -model with  $F = K^{-1}$ ,  $\phi = \phi_0 + \frac{1}{2} \ln F$ . The general  $K$ -model includes arbitrary  $u$  dependence and describes the standard plane fronted waves. It is conformal to all orders if it is conformal at leading order, i.e.  $\partial^2 K = 0$ . There exists a special scheme [1] in which a similar statement is true for the  $F$ -model, i.e. it is conformal to all orders if

$$\partial^2 F^{-1} = 0, \quad \phi = \phi_0 + \frac{1}{2} \ln F(x). \quad (2.4)$$

Perhaps the most important solution in this class is the one describing the fields outside of a fundamental string (FS) [6] which is given by

$$F^{-1} = 1 + \frac{M}{r^{D-4}}, \quad D > 4; \quad F^{-1} = 1 - M \ln \frac{r}{r_0}, \quad D = 4, \quad (2.5)$$

where  $r^2 = x_i x^i$  and  $D$  is the total number of space-time dimensions.

The key property of the  $K$ -model is that it has a covariantly constant null vector  $\partial/\partial v$ . The main features of the  $F$ -model are that there are two null Killing vectors corresponding to translations of  $u$  and  $v$ , and that the coupling to  $u, v$  is chiral (since  $G_{uv} = B_{uv}$ ). This means that the  $F$ -model is invariant under the infinite dimensional symmetry  $u' = u + f(\tau - \sigma)$  and  $v' = v + h(\tau + \sigma)$ . Associated with this symmetry are two conserved world sheet chiral currents:  $\bar{J}_u = F \bar{\partial} v$ ,  $J_v = F \partial u$ . These properties are preserved if the transverse  $x^i$ -space is modified. In fact, the two models (2.2) and (2.3) can be generalized [1] to the case when the transverse space corresponds to an arbitrary conformal  $\sigma$ -model. The simplest generalization is to keep the transverse metric flat but include an extra linear term in the dilaton.

## 2.2. The general chiral null model

The fact that a leading order solution turns out to be exact applies to a larger class of backgrounds than represented by the  $K$ -model and  $F$ -model. We will consider the following Lagrangian which will be called the chiral null model:

$$L = F(x) \partial u \bar{\partial} v + \tilde{K}(x, u) \partial u \bar{\partial} u + 2\tilde{A}_i(x, u) \partial u \bar{\partial} x^i + \partial x_i \bar{\partial} x^i + \alpha' \mathcal{R} \phi(x, u). \quad (2.6)$$

We need to assume that  $F$  does not depend on  $u$  since otherwise the argument for conformal invariance given in Section 5 does not go through. As in the case of the  $K$ -model and  $F$ -model, it is possible to replace the flat transverse space by an arbitrary conformal  $\sigma$ -model, but we will not consider that generalization here.

This model has roughly half the symmetries of the  $F$ -model. There is one null Killing vector generating shifts of  $v$ , and the action is invariant under the affine symmetry  $v' = v + h(\tau + \sigma)$  which is related to the existence of the conserved chiral current  $J_v = F(x)\partial u$ . This in turn implies the special geometrical (holonomy) properties of the corresponding string backgrounds (see Appendix A). Like the  $F$ -term, the vector coupling has a special chiral structure: the  $G_{ui}$  and  $B_{ui}$  components of the metric and the antisymmetric tensor are equal to each other.

The action (2.6) can be represented in the form

$$L = F(x)\partial u [\bar{\partial}v + K(x, u)\bar{\partial}u + 2A_i(x, u)\bar{\partial}x^i] + \partial x_i \bar{\partial}x^i + \alpha' \mathcal{R}\phi(x, u) , \quad (2.7)$$

$$K \equiv F^{-1} \tilde{K} , \quad A_i \equiv F^{-1} \tilde{A}_i ,$$

and thus is invariant under the subgroup of coordinate transformations  $v' = v - 2\eta(x, u)$  combined with a ‘gauge transformation’

$$K' = K + 2\partial_u \eta , \quad A'_i = A_i + \partial_i \eta . \quad (2.8)$$

It is clear that using this freedom one can always choose a gauge in which  $K = 0$ . However, we will often consider the special case when  $K$ ,  $A_i$  and  $\phi$  do not depend on  $u$ , i.e. when  $\partial/\partial u$  is a Killing vector. In this case,  $K$  cannot be set to zero without loss of generality.

When the fields do not depend on  $u$ , one can perform a leading-order duality transformation along any non-null direction in the  $(u, v)$ -plane. Setting  $v = \hat{v} + au$  ( $a = \text{const}$ ) in (2.7) and dualizing with respect to  $u$  yields a  $\sigma$ -model of exactly the same form with  $F$ ,  $K$ ,  $A_i$  and  $\phi$  replaced by

$$F' = (K + a)^{-1} , \quad K' = F^{-1} , \quad A'_i = A_i , \quad \phi' = \phi - \frac{1}{2} \ln[F(K + a)] . \quad (2.9)$$

In other words, chiral null models are ‘self-dual’: the null translational symmetry and chiral couplings are preserved under duality.

In Section 5 we shall determine the conditions on the functions  $F$ ,  $K$ ,  $A_i$  and  $\phi$  under which these models are conformal to all orders in  $\alpha'$ . As in the case of the simplest  $F$ -model (2.3) there exists a scheme in which these conditions turn out to be equivalent to the leading-order equations (derived in Appendix A)

$$-\frac{1}{2} \partial^2 F^{-1} + b^i \partial_i F^{-1} = 0 , \quad -\frac{1}{2} \partial_i \mathcal{F}^{ij} + b_i \mathcal{F}^{ij} = 0 , \quad (2.10)$$

$$-\frac{1}{2} \partial^2 K + b^i \partial_i K + \partial^i \partial_u A_i - 2b^i \partial_u A_i + 2F^{-1} \partial_u^2 \phi = 0 , \quad (2.11)$$

$$\phi(u, x) = \phi(u) + b_i x^i + \frac{1}{2} \ln F(x) , \quad (2.12)$$

where

$$\mathcal{F}_{ij} \equiv \partial_i A_j - \partial_j A_i , \quad \partial^2 \equiv \partial^i \partial_i .$$

Notice that the leading order equations allow a linear term  $b_i x^i$  in the dilaton. Eq. (2.12) implies that the central charge of the model is given by  $c = D + 6b^i b_i$ . One can easily verify

that these equations are invariant under the ‘gauge’ transformations (2.8) (and, when the fields do not depend on  $u$ , under the duality transformations (2.9)). When  $F, K, A_i, \phi$  are independent of  $u$  and  $b_i = 0$ , these equations take the simple form

$$\partial^2 F^{-1} = 0, \quad \partial^2 K = 0, \quad \partial_i \mathcal{F}^{ij} = 0, \quad \phi = \phi_0 + \frac{1}{2} \ln F(x). \quad (2.13)$$

A crucial feature of these equations is that they are linear. Thus all solutions satisfy a solitonic no-force condition and can be superposed (this is also true for the more general equations (2.10) - (2.12) provided  $b_i$  is held fixed). Since these equations are exact conformal invariance conditions, changing  $F, K$  or  $A_i$  while preserving (2.10)–(2.12) can be viewed as ‘marginal deformations’ of the corresponding conformal field theory.

### 2.3. Some special cases

We now discuss some special cases of the general chiral null model (2.6). If  $F = 1$ , we obtain a class of plane fronted wave backgrounds which have a covariantly constant null vector. The general background with a covariantly constant null vector contains another vector coupling [3]

$$L = \partial u \bar{\partial} v + K(x, u) \partial u \bar{\partial} u + 2A_i(x, u) \partial u \bar{\partial} x^i + 2\bar{A}_i(x, u) \partial x^i \bar{\partial} u + \partial x_i \bar{\partial} x^i + \alpha' \mathcal{R} \phi(x, u). \quad (2.14)$$

The conditions of conformal invariance of this model turn out to take the form [18] (for simplicity we set  $b_i = 0$ )

$$\partial_i \mathcal{F}^{ij} = 0, \quad \partial_i \bar{\mathcal{F}}^{ij} = 0, \quad \phi = \phi(u), \quad (2.15)$$

$$-\frac{1}{2} \partial^2 K + \partial^i \partial_u (A_i + \bar{A}_i) + \mathcal{F}^{ij} \bar{\mathcal{F}}_{ij} + 2\partial_u^2 \phi + O(\alpha'^{s+k} \partial^s \mathcal{F} \partial^k \bar{\mathcal{F}}) = 0. \quad (2.16)$$

Thus, if one breaks the chiral structure by introducing the  $\bar{A}_i$ -coupling, then, in general, there are corrections to the  $uu$ -component of the metric conformal anomaly coefficient (2.16) to all orders in  $\alpha'$ . The higher-loop corrections still vanish in one special case: when  $A_i$  and  $\bar{A}_i$  have field strengths constant in  $x$  (in general, the field strengths may still depend on  $u$ )

$$A_i = -\frac{1}{2} \mathcal{F}_{ij} x^j, \quad \bar{A}_i = -\frac{1}{2} \bar{\mathcal{F}}_{ij} x^j. \quad (2.17)$$

Such a model represents a simple and interesting conformal theory in its own right.<sup>1</sup> When the fields do not depend on  $u$  one may define the dual  $\sigma$ -model which is also conformal to all orders and will be discussed at the end of Section 5.

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<sup>1</sup> One particular case corresponds to the  $D = 4$  non-semisimple WZW model of ref. [19], namely,  $K = -x^i x_i$ ,  $A_i = -\bar{A}_i = -\frac{1}{2} \epsilon_{ij} x^j$ ,  $\phi = \text{const}$ , which is obviously a solution of (2.15), (2.16). Since  $\bar{A}_i = -A_i$ ,  $A_i$  represents the antisymmetric tensor part of the action (2.14). Another equivalent (related by a  $u$ -dependent coordinate transformation of  $x^i$ ) representation of the model of [19] is  $K = 0$ ,  $A_i = -\frac{1}{2} \epsilon_{ij} x^j$ ,  $\bar{A}_i = 0$  which will be useful at the end of Section 4 (see also Appendix A).

The special property of the model with  $\bar{A}_i = 0$  or  $A_i = 0$  (i.e. with  $G_{ui} = \pm B_{ui}$ ) resulting in cancellation of the vector-dependent contributions to the  $\beta$ -function for  $K$  was noted at the one-loop level in [18] and extended to the two-loop level in [20].<sup>2</sup> It was further shown [22] that such backgrounds are (‘half’) supersymmetric when embedded in  $D = 10$  supergravity theory and it was conjectured that these ‘supersymmetric string waves’ remain exact heterotic string solutions to all orders in  $\alpha'$  when supplemented with some gauge field background. As we shall demonstrate, (2.14) with  $\bar{A}_i = 0$  is, in fact, an exact solution of the bosonic string theory. In Section 6 we shall prove that, furthermore, it can be promoted to an exact superstring and heterotic string solution with no need to introduce an extra gauge field background. It is the chiral structure of this solution which is behind this fact.

If  $K = 0$ , and  $\tilde{A}_i(x)$ ,  $\phi$  are independent of  $u$ , the chiral null model (2.6) reduces to

$$L = F(x)\partial u\bar{\partial}v + 2\tilde{A}_i(x)\partial u\bar{\partial}x^i + \partial x_i\bar{\partial}x^i + \alpha'\mathcal{R}\phi(x) . \quad (2.18)$$

This background is also supersymmetric [23] when embedded in  $D = 10$  supergravity theory (and was also conjectured [23] to correspond to an exact heterotic string solution when supplemented by a gauge field). As above, we will prove in Section 6 that it is an exact solution of the heterotic string theory by itself, i.e. that the  $(1, 0)$  supersymmetric extension of (2.18) is a conformally invariant model without extra gauge field terms added.

#### 2.4. Examples of solutions

We now discuss some examples of solutions which are described by chiral null models. These solutions can be viewed as different generalizations of the fundamental string solution (2.5).

It is straightforward to describe the general solution for the conformal  $D = 5$  chiral null model which is independent of  $u$  (and has  $b_i = 0$ ). It is given by

$$L = F(x)\partial u [\bar{\partial}v + K(x)\bar{\partial}u + 2A_i(x)\bar{\partial}x^i] + \partial x_i\bar{\partial}x^i + \alpha'\mathcal{R}\phi(x) , \quad (2.19)$$

where the functions  $F$ ,  $K$ ,  $A_i$  and  $\phi$  satisfy (2.13). Since the transverse space is now three dimensional, every solution  $A_i$  to Maxwell’s equation can be written in terms of a scalar<sup>3</sup>

$$\epsilon^{ijk}\partial_j A_k = \partial^i T(x) , \quad \partial^2 T = 0 . \quad (2.20)$$

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<sup>2</sup> It was observed in [20] that introducing the generalized connection with the antisymmetric tensor field strength as torsion, one finds that if  $\bar{A}_i = 0$  the generalized curvature (see Appendix A) is nearly flat: the only non-trivial components of it are  $\hat{R}_{-ijk}^v = 2\partial_i \mathcal{F}_{jk}$ ,  $\hat{R}_{-iuj}^v = 2\partial_i \partial_u A_j - \partial_i \partial_j K$ . Then assuming that all terms in the  $\beta_{\mu\nu}$ -function have the structure  $Y_\mu^{\lambda\rho\sigma} \hat{R}_{-\lambda\rho\sigma\nu}$ , where  $Y$  depends on  $H_{\mu\nu\lambda}$  and  $R_{\mu\nu\lambda\rho}$  (in a special renormalization scheme this is true at the 2-loop order [21]) one can argue [20] that all higher-order corrections vanish. This argument is not completely rigorous and, in fact, unnecessary, since a simpler direct proof of conformal invariance of this model can be given (see Section 5).

<sup>3</sup> Another simple case is  $D = 4$  since in two transverse dimensions  $A_i = q\epsilon_{ij}x^j$ .

With  $F^{-1}$  and  $K$  also satisfying Laplace's equation in the transverse space, the general solution is characterized by three harmonic functions. It is clear from (2.11) that the model remains conformal if we let  $K$  have an arbitrary  $u$  dependence. If we set  $A_i = 0$ , take  $F$  and  $\phi$  given by the FS solution (2.5), and keep  $K$  general, the solutions describe traveling waves along the fundamental string and were first discussed in [7].

Consider now spherically symmetric solutions with  $A_i = 0$  and no  $u$  dependence. Since all spherically symmetric solutions to Laplace's equation take the form  $a + br^{4-D}$ , the function  $K$  can always be represented as  $K(x) = c + nF^{-1}(x)$ . After a shift of  $v$  the model then takes the form (2.6) with  $\tilde{K} = n$ . In view of the freedom to rescale  $u$  and  $v$  the only non-trivial values of the constant  $n$  are 0 and 1.  $n = 0$  corresponds the standard FS while  $n = 1$  yields the following simple generalization

$$L = F(x)\partial u\bar{\partial}v + \partial u\bar{\partial}u + \partial x_i\bar{\partial}x^i + \alpha'\mathcal{R}\phi(x) , \quad (2.21)$$

where  $F$  and  $\phi$  are given by (2.5) and (2.4). This solution was first found in [9] and further discussed in [24].

It is known [25] that the fundamental string is the extremal limit of a family of charged black string solutions to the leading order equations. The generalization (2.21) can similarly be viewed as the extremal limit of a black string as follows (we consider  $D = 5$  for simplicity). The charged black string can be obtained by boosting the direct product of the Schwarzschild background with a line, and applying a duality transformation [26]. The result is ( $S \equiv \sinh \alpha$ ,  $C \equiv \cosh \alpha$ ,  $\alpha$  is the original boost parameter)

$$ds^2 = \left(1 + \frac{2mS^2}{r}\right)^{-1} \left[ - \left(1 - \frac{2m}{r}\right) dt^2 + dy^2 \right] + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega , \quad (2.22)$$

$$B_{yt} = \frac{C}{S} \left(1 + \frac{2mS^2}{r}\right)^{-1} , \quad e^{-2\phi} = 1 + \frac{2mS^2}{r} .$$

The extremal limit corresponds to sending  $m \rightarrow 0$ ,  $\alpha \rightarrow \infty$  in such a way that  $M \equiv 2me^{2\alpha}$  is held fixed. In this limit the horizon at  $r = 2m$  shrinks down to zero size and becomes singular. The charged black string solution (2.22) approaches the fundamental string (2.3). If we add linear momentum to (2.22) by applying a boost  $t = \hat{t} \cosh \beta + \hat{y} \sinh \beta$ ,  $y = \hat{t} \sinh \beta + \hat{y} \cosh \beta$ , and then take the extremal limit  $m \rightarrow 0$ ,  $\alpha, \beta \rightarrow \infty$  with  $M \equiv 2me^{2\alpha} = 2me^{2\beta}$  fixed, we obtain the generalized fundamental string (2.21). So this solution can also be viewed as the extremal limit of a charged black string, but now with a non-zero linear momentum.

### 3. Dimensional reduction

To consider further applications of the chiral null models to, for example, extremal dilatonic black holes in  $D = 4$  and charged FS solutions, we need to discuss first the Kaluza-Klein re-interpretation of higher dimensional bosonic string solutions (heterotic string solutions will be discussed in Section 6). To have extremal black holes we need gauge fields. There are no fundamental gauge fields in bosonic string theory but they appear once the theory is compactified on a torus or a group manifold and is expressed in terms of 'lower-dimensional' geometrical objects.



The usual treatment of dimensional reduction in field theory starts with a spacetime action. This is possible also in string theory, but difficult to do exactly. One would have to start with the full massless string effective action in, say, five dimensions containing terms of all orders in  $\alpha'$ . Assuming the fifth direction  $x^5$  is periodic we can expand the metric, antisymmetric tensor and dilaton in Fourier series in  $x^5$  and explicitly integrate over  $x^5$ . The result will be the effective action in  $D = 4$  containing massless fields as well as an infinite tower of massive modes with masses proportional to a compactification scale. Any exact solution of the  $D = 5$  theory which does not depend on  $x^5$  can then be directly interpreted as a solution of the equations of the  $D = 4$  ‘compactified’ theory with all massive modes set equal to zero (but all ‘massless’  $\alpha'$ -terms included).

Fortunately, in string theory there is a simpler alternative – to perform the dimensional reduction directly at the more fundamental level of the string action itself. Let us start with the general string  $\sigma$ -model (2.1), split the coordinates  $X^M$  into ‘external’  $x^\mu$  and ‘internal’  $y^a$  and assume that the couplings do not depend on  $y^a$ ,

$$L = (G_{\mu\nu} + B_{\mu\nu})(x)\partial x^\mu \bar{\partial} x^\nu + (\mathcal{A}_{\mu a} + \mathcal{B}_{\mu a})(x)\partial x^\mu \bar{\partial} y^a + (\mathcal{A}_{\mu a} - \mathcal{B}_{\mu a})(x)\bar{\partial} x^\mu \partial y^a \quad (3.1) \\ + (G_{ab} + B_{ab})(x)\partial y^a \bar{\partial} y^b + \alpha' \mathcal{R}\phi(x) ,$$

where

$$\mathcal{A}_{\mu a} \equiv G_{\mu a} , \quad \mathcal{B}_{\mu a} \equiv B_{\mu a} . \quad (3.2)$$

Assuming for simplicity that  $B_{ab} = 0$ , it is easy to represent the action in a form which is manifestly invariant under the space-time gauge transformations of the vector fields  $\mathcal{A}_\mu^a \equiv G^{ab} \mathcal{A}_{\mu b}$  and  $\mathcal{B}_{\mu a}$

$$L = (\hat{G}_{\mu\nu} + B_{\mu\nu})(x)\partial x^\mu \bar{\partial} x^\nu + \mathcal{B}_{\mu a}(x)(\partial x^\mu \bar{\partial} y^a - \bar{\partial} x^\mu \partial y^a) \quad (3.3) \\ + G_{ab}(x) [\partial y^a + \mathcal{A}_\mu^a(x)\partial x^\mu] [\bar{\partial} y^b + \mathcal{A}_\nu^b(x)\bar{\partial} x^\nu] + \alpha' \mathcal{R}\phi(x) ,$$

where the gauge-invariant ‘Kaluza-Klein’ metric  $\hat{G}_{\mu\nu}$  is defined by

$$\hat{G}_{\mu\nu} \equiv G_{\mu\nu} - G_{ab} \mathcal{A}_\mu^a \mathcal{A}_\nu^b . \quad (3.4)$$

Like all  $\sigma$ -model Lagrangians, (3.3) changes by a total derivative if one adds the curl of a vector to the antisymmetric tensor field. Since we are assuming no dependence on  $y^a$ , the  $(\mu, a)$ -component of this transformation is simply  $\mathcal{B}_{\mu a} \rightarrow \mathcal{B}_{\mu a} + \partial_\mu \lambda_a$ , i.e. the standard gauge transformation for the vector fields  $\mathcal{B}_{\mu a}$ . The action (3.3) is also invariant under shifting  $y^a \rightarrow y^a - \eta^a(x)$  together with

$$\mathcal{A}_\mu^a \rightarrow \mathcal{A}_\mu^a + \partial_\mu \eta^a , \quad B_{\mu\nu} \rightarrow B_{\mu\nu} - 2\partial_{[\mu} \eta^a \mathcal{B}_{\nu]a} . \quad (3.5)$$

The first transformation is the usual one for the vector fields  $\mathcal{A}_\mu^a$  while the second implies that the gauge-invariant antisymmetric tensor field strength is given by<sup>4</sup>

$$\hat{H}_{\lambda\mu\nu} = 3\partial_{[\lambda} B_{\mu\nu]} - 3\mathcal{A}_{[\lambda}^a \mathcal{B}_{\mu\nu]a} , \quad \mathcal{B}_{\mu\nu a} \equiv 2\partial_{[\mu} \mathcal{B}_{\nu]a} . \quad (3.6)$$

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<sup>4</sup> From the world sheet point of view we are using there seems to be no reason to redefine the antisymmetric tensor  $B_{\mu\nu}$  in (3.3) by the term  $\mathcal{A}_{[\mu}^a \mathcal{B}_{\nu]a}$  as it is sometimes done in the effective action approach to dimensional reduction. If one does such a redefinition, the new  $\hat{B}_{\mu\nu}$  also transforms under the  $\mathcal{B}_{\mu a}$  gauge transformations and the generalized field strength tensor  $\hat{H}_{\lambda\mu\nu}$  takes a more ‘symmetric’ form with respect to the two vector fields  $\mathcal{A}_\mu^a$  and  $\mathcal{B}_{\mu a}$ . It should be noted, however, that it is the full  $\hat{H}_{\lambda\mu\nu}$  that has an invariant meaning, and it remains the same irrespective of the definition of  $\hat{B}_{\mu\nu}$ .

Although the world sheet approach to dimensional reduction in string theory is the most straightforward and simplest, it is useful to recall what the corresponding procedure looks like from the point of view of the space-time effective action. For example, if we start with just the leading-order term in the  $D = 5$  bosonic string action

$$S_5 = \kappa_0 \int d^5x \sqrt{G} e^{-2\phi} \{ R + 4(\partial_M \phi)^2 - \frac{1}{12}(H_{MNK})^2 + O(\alpha') \} , \quad (3.7)$$

and assume that all the fields are independent of  $x^5$ , we obtain the four dimensional reduced action (for the general case, see e.g. [27] and refs. there)

$$S_4 = \hat{\kappa}_0 \int d^4x \sqrt{\hat{G}} e^{-2\phi+\sigma} \{ \hat{R} + 4(\partial_\mu \phi)^2 - 4\partial_\mu \phi \partial^\mu \sigma - \frac{1}{12}(\hat{H}_{\mu\nu\lambda})^2 - \frac{1}{4}e^{2\sigma}(\mathcal{F}_{\mu\nu})^2 - \frac{1}{4}e^{-2\sigma}(\mathcal{B}_{\mu\nu})^2 + O(\alpha') \} , \quad (3.8)$$

where we have defined

$$G_{55} \equiv e^{2\sigma}, \quad \mathcal{F}_{\mu\nu} = 2\partial_{[\mu}\mathcal{A}_{\nu]}, \quad \mathcal{B}_{\mu\nu} = 2\partial_{[\mu}\mathcal{B}_{\nu]}, \quad \mathcal{A}_\mu \equiv \mathcal{A}_\mu^5, \quad \mathcal{B}_\mu \equiv \mathcal{B}_{\mu 5} . \quad (3.9)$$

Setting

$$\varphi = 2\phi - \sigma \quad (3.10)$$

the action (3.8) becomes

$$S_4 = \hat{\kappa}_0 \int d^4x \sqrt{\hat{G}} e^{-\varphi} \{ \hat{R} + (\partial_\mu \varphi)^2 - (\partial_\mu \sigma)^2 - \frac{1}{12}(\hat{H}_{\mu\nu\lambda})^2 - \frac{1}{4}e^{2\sigma}(\mathcal{F}_{\mu\nu})^2 - \frac{1}{4}e^{-2\sigma}(\mathcal{B}_{\mu\nu})^2 + O(\alpha') \} . \quad (3.11)$$

In the Einstein frame (3.11) takes the form

$$S_4 = \hat{\kappa}_0 \int d^4x \sqrt{\hat{G}_E} \{ \hat{R}_E - \frac{1}{2}(\partial_\mu \varphi)^2 - (\partial_\mu \sigma)^2 - \frac{1}{12}e^{-2\varphi}(\hat{H}_{\mu\nu\lambda})^2 - \frac{1}{4}e^{-\varphi+2\sigma}(\mathcal{F}_{\mu\nu})^2 - \frac{1}{4}e^{-\varphi-2\sigma}(\mathcal{B}_{\mu\nu})^2 + O(\alpha') \} . \quad (3.12)$$

Thus, in general, the four dimensional theory contains two scalars, two vectors, and the antisymmetric tensor, as well as the metric. In certain special cases, the nontrivial part of the action (3.12) can be expressed in terms of only one scalar and one vector, so that it takes the familiar form<sup>5</sup>

$$S_4 = \hat{\kappa}_0 \int d^4x \sqrt{\hat{G}_E} \{ \hat{R}_E - \frac{1}{2}(\partial_\mu \psi)^2 - \frac{1}{4}e^{-a\psi}(\mathcal{F}_{\mu\nu})^2 + O(\alpha') \} . \quad (3.13)$$

For example, if one sets  $\phi = 0$  and  $H_{MNK} = 0$  in the  $D = 5$  action, or equivalently  $\varphi = -\sigma$ ,  $\hat{H}_{\mu\nu\lambda} = 0 = \mathcal{B}_{\mu\nu}$  directly in (3.12), one obtains (3.13) with  $\psi = -a\sigma$  and  $a = \sqrt{3}$ . This is, of course, the standard Kaluza-Klein reduction of the Einstein action. Another possibility is to set  $\sigma = 0$  ( $G_{55} = 1$ ),  $\hat{H}_{\mu\nu\lambda} = 0$  and either the two vector fields proportional to each other, or let one of them vanish. This case corresponds to (3.13) with  $\psi = \varphi$  and  $a = 1$ .

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<sup>5</sup> Such ansatzes must, of course, be consistent with  $D = 5$  equations of motion.

## 4. Solutions involving dimensional reduction

In this section we discuss the dimensional reduction of some of the exact solutions described by chiral null models (2.6). We will see that several previously found solutions of the leading order string effective equations can be easily obtained in this way. In addition, we find some solutions which appear to be new.

### 4.1. Charged fundamental string solutions

Our first example is the charged FS solution found at the leading order level in [8,9].<sup>6</sup> This solution is obtained by starting with the general chiral null model in  $D+N$  dimensions, and requiring that all fields be independent of  $u$  and  $N$  of the transverse dimensions labeled by  $y^a$ . If we further assume that the vector coupling has only  $y^a$ -components, we obtain

$$L = F(x)\partial u\bar{\partial}v + \tilde{K}(x)\partial u\bar{\partial}u + \partial x_i\bar{\partial}x^i + 2\tilde{A}_a(x)\partial u\bar{\partial}y^a + \partial y_a\bar{\partial}y^a + \alpha'\mathcal{R}\phi(x) , \quad (4.1)$$

which is conformal to all orders provided  $F$ ,  $K \equiv F^{-1}\tilde{K}$ ,  $A_a \equiv F^{-1}\tilde{A}_a$  and  $\phi$  satisfy (2.13). If we are looking for FS-type solutions which are rotationally symmetric in  $D-2$  coordinates  $x^i$ , then solving the Laplace equations we can put the functions  $F, K, A_a$  in the form<sup>7</sup>

$$F^{-1} = 1 + \frac{M}{r^{D-4}} , \quad \phi = \phi_0 + \frac{1}{2}\ln F(r) , \quad r^2 = x^i x_i ,$$

$$K = c + \frac{P}{r^{D-4}} , \quad A_a = \frac{Q_a}{r^{D-4}} . \quad (4.2)$$

Shifting  $v$  we can thus in general replace  $\tilde{K}$  in (4.1) by a constant. To re-interpret (4.1) as a  $D$ -dimensional model coupled to  $N$  internal coordinates we rewrite it in the form (3.3)

$$L = F(r)\partial u\bar{\partial}v + \tilde{K}'(r)\partial u\bar{\partial}u + \partial x_i\bar{\partial}x^i + \alpha'\mathcal{R}\phi(r)$$

$$+ \tilde{A}_a(r)(\partial u\bar{\partial}y^a - \partial y^a\bar{\partial}u) + [\partial y^a + \tilde{A}^a(r)\partial u][\bar{\partial}y_a + \tilde{A}_a(r)\bar{\partial}u] ,$$

$$\tilde{K}'(r) \equiv \tilde{K} - (\tilde{A}_a)^2 .$$

The first four terms give the  $D$ -dimensional space-time metric, antisymmetric tensor and dilaton while the last two identify (see (3.3)) the presence of two equal vector field backgrounds (two equal components  $G_{ua}$  and  $B_{ua}$  conspire as one  $D$ -dimensional Kaluza-Klein vector field, cf. (3.11)). Note that since  $G_{ab} = \delta_{ab}$ , the modulus field is constant and the lower dimensional dilaton is the same as the higher dimensional one. In the case of just one internal dimension we get one abelian vector field  $u$ -component and the resulting background becomes that of the charged FS in [8,9].

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<sup>6</sup> The method of [8] was to start with the neutral solution and to make the most general leading order duality rotation in all available isometric directions (including the internal ones). Since the duality transformation has, in general,  $\alpha'$ -corrections, this procedure does not guarantee the exactness of the resulting solution.

<sup>7</sup> In the zero charge  $Q_a = 0$  limit we get not just the FS solution of [6] but its modification (2.21) which corresponds to momentum running along the string.

#### 4.2. $D = 4$ solutions with electromagnetic fields

To obtain four dimensional solutions with electromagnetic fields, we can reduce a  $D = 5$  chiral null model. It was recently shown [2] that extremal electrically charged black holes can be obtained in this way. If one starts with the standard  $D = 5$  FS (2.3),(2.5) one gets [28] the extremal electric black hole solution to (3.13) with  $a = \sqrt{3}$  which was discussed in [10], while starting with the generalized FS (2.21) one obtains the extremal electric black hole solution to (3.13) with  $a = 1$  discussed in [11,12].<sup>8</sup>

Here we shall consider the most general  $D = 5$  chiral null model which is independent of  $u$ . It will yield a large class of  $D = 4$  solutions. Some of these backgrounds were recently found [13,14,15] as leading-order string solutions, i.e. solutions of the dilaton-axion generalization of the  $D = 4$  Einstein-Maxwell theory. They are the analogs of the IWP (Israel-Wilson-Perjés [29]) solution of the pure Einstein-Maxwell theory.<sup>9</sup> Special cases of this generalized IWP solution describe a collection of extremal electric dilatonic black holes (Majumdar-Papapetrou-type solution) and an extremal electric Taub-NUT-type solution.

The  $D = 5$  chiral null model which is independent of  $u$

$$L_5 = F(x)\partial u [\bar{\partial}v + K(x)\bar{\partial}u + 2A_i(x)\bar{\partial}x^i] + \partial x_i \bar{\partial}x^i + \alpha' \mathcal{R}\phi(x) , \quad (4.4)$$

was discussed in section 2.4 where it was noted that the general solution depends on the three harmonic functions  $F^{-1}$ ,  $K$  and  $T$  (see (2.20)) of the three coordinates  $x^i$ . This model can be reduced to  $D = 4$  along any space-like direction in the  $u, v$  plane. Shifting  $v$  by a multiple of  $u$  changes, of course, the direction of  $\partial/\partial u$ , but this transformation is equivalent to a shift of  $K$  by a constant. Shifting  $u$  by a multiple of  $v$  can be undone by a particular case of the gauge transformation (3.5) (which gives an equivalent background, in particular, leaves  $\hat{H}_{\mu\nu\lambda}$  invariant). Thus it suffices to use  $u$  as the internal coordinate  $y$  (which is possible, provided  $FK > 0$ ) and to identify  $v$  with  $2t$ . Then we can put (4.4) in the “four-dimensional” form (3.3) as follows

$$\begin{aligned} L_5 = & -K(x)^{-1}F(x) [\partial t + A_i(x)\partial x^i] [\bar{\partial}t + A_i(x)\bar{\partial}x^i] + \partial x_i \bar{\partial}x^i + \alpha' \mathcal{R}\phi(x) \\ & + F(x)(\partial y \bar{\partial}t - \partial t \bar{\partial}y) + F(x)A_i(x)(\partial y \bar{\partial}x^i - \partial x^i \bar{\partial}y) \\ & + K(x)F(x) [\partial y + K^{-1}(x)\partial t + K^{-1}(x)A_i(x)\partial x^i] \\ & \times [\bar{\partial}y + K^{-1}(x)\bar{\partial}t + K^{-1}(x)A_i(x)\bar{\partial}x^i] . \end{aligned} \quad (4.5)$$

The corresponding *four-dimensional* background is thus represented by the following metric, two abelian gauge fields  $\mathcal{A}_\mu^5 \equiv \mathcal{A}_\mu$ ,  $\mathcal{B}_{\mu 5} \equiv \mathcal{B}_\mu$ , two scalars (the ‘modulus’  $\sigma = \frac{1}{2}\ln G_{55}$  and the dilaton) and the antisymmetric tensor field strength  $\hat{H}$  (cf. (3.3), (3.11))

$$ds^2 = -F(x)K^{-1}(x) [dt + A_i(x)dx^i]^2 + dx_i dx^i , \quad (4.6)$$

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<sup>8</sup> The  $a = \sqrt{3}$  black hole can also be obtained [10] from the  $D = 5$  plane-wave-type background (2.2) which is dual to FS. Similarly, one can get the  $a = 1$  electric dilatonic  $D = 4$  black hole from a duality-rotated (2.9) version of the generalized FS (2.21). Such model is, however, essentially equivalent to (2.21), since it is ‘self-dual’.

<sup>9</sup> It was shown also that these backgrounds are supersymmetric when embedded in a supergravity [13].

$$\begin{aligned}\mathcal{A}_t &= K^{-1}(x) , \quad \mathcal{A}_i = K^{-1}(x)A_i(x) , \quad \mathcal{B}_t = -F(x) , \quad \mathcal{B}_i = -F(x)A_i(x) , \\ \sigma &= \frac{1}{2}\ln[F(x)K(x)] , \quad \phi = \phi_0 + \frac{1}{2}\ln F(x) , \quad \hat{H}_{\lambda\mu\nu} = -6\mathcal{A}_{[\lambda}\partial_\mu\mathcal{B}_{\nu]} .\end{aligned}$$

Notice that even though the  $D = 4$  antisymmetric tensor  $B_{\mu\nu}$  vanishes, the gauge invariant field strength  $\hat{H}_{\lambda\mu\nu}$  is nonzero due to the contribution from the gauge fields in (3.6). This background represents a solution of the equations following from the  $D = 4$  effective action (3.11) since  $A_i$  satisfies  $\epsilon^{ijk}\partial_j A_k = \partial^i T(x)$ , and  $F^{-1}, K$  and  $T$  are solutions of the three dimensional Laplace equation.

Let us now consider some special cases. If  $K = 1$  and  $A_i = 0$ , the gauge field  $\mathcal{A}_\mu$  becomes trivial and the two scalars coincide (up to a constant). Since the gauge fields have only time components being nonzero, the antisymmetric tensor  $\hat{H}$  vanishes. If we now set  $F^{-1} = 1 + M/r$ , the original  $D = 5$  theory (4.4) describes the fundamental string and the  $D = 4$  reduction is the ‘Kaluza-Klein’ extremal black hole, i.e. the extreme electrically charged black hole solution corresponding to (3.13) with  $a = \sqrt{3}$ . We see that this solution has a straightforward generalization to the case of  $A_i \neq 0$ .

The case  $K = F^{-1}$  is of particular interest. The  $D = 5$  model (4.4) is the  $A_i$ -generalization of (2.21) while the corresponding  $D = 4$  background is

$$\begin{aligned}ds^2 &= -F^2(x) [dt + A_i(x)dx^i]^2 + dx_i dx^i , \\ \mathcal{A}_t &= F(x) , \quad \mathcal{A}_i = F(x)A_i(x) , \quad \mathcal{B}_\mu = -\mathcal{A}_\mu , \\ \phi &= \phi_0 + \frac{1}{2}\ln F(x) , \quad \hat{H}_{\lambda\mu\nu} = 6\mathcal{A}_{[\lambda}\partial_\mu\mathcal{A}_{\nu]} , \quad \sigma = 0 .\end{aligned}\tag{4.7}$$

Since  $\sigma = 0$  and the two gauge fields differ only by a sign, these backgrounds are solutions to (3.13) with  $a = 1$  provided the antisymmetric tensor term of (3.12) is included. These are precisely the  $D = 4$  dilatonic IWP solutions [13,14,15]. If we restrict further to  $A_i = 0$  and  $F^{-1} = 1 + M/r$ , then  $\hat{H}_{\lambda\mu\nu} = 0$  and we obtain the ‘standard’ extremal dilatonic black hole [11,12]<sup>10</sup>

$$ds^2 = -F^2(r)dt^2 + dx_i dx^i ,\tag{4.8}$$

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<sup>10</sup> Let us note that the  $D = 4$  extremal electric dilatonic black hole background can also be related to a  $D = 6$  chiral null model with  $K = 0$ ,  $L_6 = F(x)\partial u [\bar{\partial}v + 2A(x)\bar{\partial}y'] - \partial y' \bar{\partial} y' + \partial x_i \bar{\partial} x^i$ , where the internal coordinate  $y'$  has the ‘wrong’ (time-like) signature. Introducing the new coordinate  $y' = y + u$  and choosing  $A = F^{-1}$  (which is consistent with the conformal invariance conditions) we find that this model takes the form of (2.21) plus an extra free time-like direction,  $L_6 = F(x)\partial u \bar{\partial} v + \partial u \bar{\partial} u + \partial x_i \bar{\partial} x^i - \partial y \bar{\partial} y$ , and thus can also be related to the  $D = 4$  extremal electric black hole. An equivalent observation was made at the level of the leading-order terms in the effective action in [30] (ref. [13] also discussed a similar higher (six) dimensional interpretation of the IWP solution). It should be emphasized that it is our  $D = 5$  model (4.4) that provides the correct higher-dimensional embedding of these  $D = 4$  black-hole type solutions: though the presence of an extra time-like ‘internal’ coordinate in the above  $D = 6$  model is irrelevant from the point of view of the proof of exactness of the  $D = 4$  solution, it is unphysical, since complex coordinate transformations are needed if one wants to keep the physical signature of the full higher-dimensional space.

$$\mathcal{A}_t = -\mathcal{B}_t = F(r) , \quad \phi(x) = \phi_0 + \frac{1}{2} \ln F(r) , \quad \mathcal{A}_i = \mathcal{B}_i = \sigma = \hat{H}_{\lambda\mu\nu} = 0 .$$

Adding a nonzero  $A_i$  to this solution by setting  $T = q/r$  has the effect of adding a NUT charge. The result is the extremal electrically charged dilatonic Taub-NUT solution. Linear superposition of an arbitrary number of solutions of this type is possible by setting

$$F^{-1} = 1 + \sum_{k=1}^N \frac{M_k}{|x - x_k|} , \quad T = \sum_{k=1}^N \frac{q_k}{|x - x_k|} . \quad (4.9)$$

To add angular momentum, one takes solutions to Laplace's equation which are singular on circles, rather than points as in (4.9).

Finally, if we set  $K = F = 1$  in (4.7), the dilaton becomes constant. This solution depends only on  $A_i$  and describes a spacetime with a magnetic field  $\mathcal{F}_{ij} = 2\partial_{[i}A_{j]}$  and antisymmetric tensor  $\hat{H}_{tij} = \mathcal{F}_{ij}$ . The corresponding  $D = 5$  exact conformal  $\sigma$ -model (4.4) can be put (by a shift of  $v$ ) in the following simple form

$$L = \partial u \bar{\partial} v + 2A_i(x) \partial u \bar{\partial} x^i + \partial x_i \bar{\partial} x^i , \quad \partial_i \mathcal{F}^{ij} = 0 , \quad (4.10)$$

and deserves further study. Some special choices of  $A_i$  are particularly interesting. One is the monopole background,  $\mathcal{F}_{ij} = q\epsilon_{ijk}x^k/|x|^3$ . Another is the case of a uniform magnetic field,  $\mathcal{F}_{ij} = \text{const}$ , i.e.  $A_i = -\frac{1}{2}\mathcal{F}_{ij}x^j$ . This model is equivalent (see Appendix A.3) to a product of the non-semisimple  $D = 4$  WZW model of [19] and an extra free spatial direction and thus has a CFT interpretation. One can choose coordinates so that the  $D = 4$  metric for the uniform magnetic field solution is simply

$$ds^2 = - \left( dt + \frac{1}{2} \mathcal{H} r^2 d\theta \right)^2 + dz^2 + dr^2 + r^2 d\theta^2 , \quad (4.11)$$

and describes a rotating universe (while the antisymmetric tensor  $\hat{H}$  is constant). This uniform magnetic field solution may be contrasted with the dilatonic Melvin solution [11,31] in which the magnetic field decreases with transverse distance. The latter solution contains a nonconstant dilaton (but no antisymmetric tensor or rotation) and is expected to have higher order  $\alpha'$  corrections.

The solutions (4.6) with generic  $K$  and thus different gauge fields  $\mathcal{A}_i$  and  $\mathcal{B}_i$  appear to be new.

## 5. Conformal invariance of the chiral null models

The aim of this section is to demonstrate that the general chiral null model (2.6) is conformal to all orders in  $\alpha'$  provided the couplings satisfy the conditions (2.10) - (2.12) and one choses a special renormalization scheme. Our discussion will be based on the approach of [1] where more details about the special choice of the scheme can be found.

In [1] it was shown that the  $F$ -model (2.3) (i.e. (2.6) with  $K = A_i = 0$ ) which has two null Killing vectors and two associated chiral currents, is exact. It turns out that a single chiral current associated with a null symmetry is, in fact, sufficient to establish the exact conformal invariance of the more general backgrounds (2.6).

To find the conditions for conformal invariance of a  $\sigma$ -model we must define it on a curved two dimensional surface, introduce sources for the  $\sigma$ -model fields and determine when the resulting generating functional (or its Legendre transform) does not depend on the conformal factor of the 2-metric. There are two reasons why the models (2.6) are special. First, the null symmetry and chiral coupling to  $v$  imply that the path integral over  $v$  is readily computable giving a  $\delta$ -function constraint on  $u$  which expresses  $u$  in terms of  $x^i$  and a source. Second, chirality of the  $\partial u \partial x$ -coupling implies that the resulting effective  $x$ -theory has only tadpole divergences (or conformal anomalies) in a properly chosen scheme.

We shall first give the proof of conformal invariance in a few special cases mentioned in section 2 (when some of the functions in (2.6) are trivial) and then give the general argument.

### 5.1. $F=1$

The argument is simplest when  $F = 1$ . To find the exact conditions of conformal invariance we follow [1] by adding the source terms ( $z$  denotes the two world sheet coordinates)

$$L_{source} = V(z) \partial \bar{\partial} u + U(z) \partial \bar{\partial} v + X_i(z) \partial \bar{\partial} x^i , \quad (5.1)$$

to (2.6) and performing the path integral over  $v$ . The resulting  $\delta$ -function sets  $u$  to its classical value  $U$  (up to a zero mode which we absorb in  $U$ ). Thus  $u$  is ‘frozen’ and the effective  $x$ -theory is

$$\begin{aligned} L_{eff} = & \partial x_i \bar{\partial} x^i + K(x, U) \partial U \bar{\partial} U + 2A_i(x, U) \partial U \bar{\partial} x^i + \alpha' \mathcal{R} \phi(x, U) \\ & + X_i \partial \bar{\partial} x^i + V \partial \bar{\partial} U . \end{aligned} \quad (5.2)$$

Computing the classical dilaton contribution ( $\sim \partial \bar{\partial} \phi$ ) to the trace of the stress energy tensor and observing that there cannot be  $O(\partial U \partial x^i)$  quantum contributions (in view of the absence of the  $O(\bar{\partial} U)$  vector coupling and simple dimensional considerations) one finds that the necessary conditions for this theory to be conformal are  $\partial_i \partial_u \phi = 0$ ,  $\partial_i \partial_j \phi = 0$ , so that

$$\phi(x, u) = \phi(u) + b_i x^i , \quad b_i = \text{const} . \quad (5.3)$$

One also learns that (in the minimal subtraction scheme) the renormalization of the  $\partial U \bar{\partial} U$  and  $\partial U \bar{\partial} X^i$  may come only from the one-loop tadpole diagrams. The conclusion is that this model is conformal to all orders once the leading-order conditions of conformal invariance are satisfied (see also [18])

$$-\frac{1}{2} \partial^2 K + b^i \partial_i K + \partial^i \partial_u A_i - 2b^i \partial_u A_i + 2\partial_u^2 \phi = 0 , \quad -\frac{1}{2} \partial_i \mathcal{F}^{ij} + b_i \mathcal{F}^{ij} = 0 . \quad (5.4)$$

These relations follow from a direct computation of the tadpole graphs and use of classical  $\sigma$ -model equations to transform the dilaton contribution (for simplicity, one may gauge away  $K$  by using the freedom (2.8)). They agree, of course, with the standard general expression for the one loop Weyl anomaly coefficients given in Appendix A.

## 5.2. $A_i = 0$

Let us now set  $A_i = 0$  and assume that  $K = F^{-1}\tilde{K}$  and  $\phi$  do not depend on  $u$ , i.e. consider

$$L = F(x)\partial u\bar{\partial}v + \tilde{K}(x)\partial u\bar{\partial}u + \partial x_i\bar{\partial}x^i + \alpha'\mathcal{R}\phi(x) . \quad (5.5)$$

Introducing the source terms (5.1) and integrating over  $v$  one finds the constraint

$$\partial u = F^{-1}(x)\partial U . \quad (5.6)$$

Integrating then over  $u$  and taking into account the determinant contribution that shifts the dilaton as well as fixing the same special ‘leading-order’ scheme (related to the standard one by an  $\alpha'$ -redefinition of the  $ij$ -component of the metric) as in the  $F$ -model [1] one finds that the effective  $x$ -theory takes the form<sup>11</sup>

$$L_{eff} = \partial x_i\bar{\partial}x^i - F^{-1}(x)\partial U\bar{\partial}V + K(x)\partial U\bar{\partial}\partial^{-1}[F^{-1}(x)\partial U] \quad (5.7)$$

$$+ \alpha'\mathcal{R}\phi'(x) + X_i\partial\bar{\partial}x^i ,$$

$$\phi' \equiv \phi - \frac{1}{2}\ln F(x) . \quad (5.8)$$

The conditions of exact conformal invariance include the linearity of the dilaton  $\phi'$  in  $x$

$$\phi' = \phi_0 + b_i x^i , \quad \phi = \phi_0 + b_i x^i + \frac{1}{2}\ln F , \quad (5.9)$$

and the standard scalar (‘tachyonic’) equation for  $F^{-1}$

$$-\frac{1}{2}\partial^2 F^{-1} + b^i\partial_i F^{-1} = 0 . \quad (5.10)$$

The conformal anomaly must be local, so it is only the local part of the quantum average of the non-local  $O(\partial U\bar{\partial}U)$  term that may contribute to it. Since this non-local term already contains two factors of  $\partial U$  it cannot produce  $\partial x$ -dependent counterterms. That means we may expand the functions  $K(x)$  and  $F^{-1}(x)$  in it near a constant,  $x^i(z) = x_0^i + \eta^i(z)$ ,

$$\begin{aligned} & \int d^2z d^2z' [K(x)\partial U](z)\bar{\partial}^2\Delta^{-1}(z, z')[F^{-1}(x)\partial U](z') \\ &= \sum_{n,m=0}^{\infty} \frac{1}{n!m!} \partial_{i_1}\dots\partial_{i_m} K(x_0)\partial_{j_1}\dots\partial_{j_n} F^{-1}(x_0) \\ & \int d^2z d^2z' (\eta^{i_1}\dots\eta^{i_m})(z)\partial U(z)\bar{\partial}^2\Delta^{-1}(z, z')(\eta^{j_1}\dots\eta^{j_n})(z')\partial U(z') , \end{aligned} \quad (5.11)$$

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<sup>11</sup> Note that if  $F$  were  $u$ -dependent the integral over  $u$  would not be easily computable and the argument below would not apply.



where we defined  $\Delta^{-1}$  by  $\partial\bar{\partial}\Delta^{-1} = \delta^{(2)}(z, z')$ . Then the only contractions of the quantum fields  $\eta^i$  that can produce local  $O(\partial U \bar{\partial} U)$  divergences are the one-loop tadpoles on the left and right side of the non-local propagator  $\Delta^{-1}(z, z')$ . Any contraction between  $\eta^n(z)$  and  $\eta^m(z')$  gives additional  $\Delta^{-1}(z, z')$ -factor and thus contributes only to the non-local part of the corresponding  $2d$  effective action.

As a result, we find the following conformal invariance condition

$$F^{-1}\partial^2 K + K\partial^2 F^{-1} = 2b^i F^{-1}\partial_i K + 2b^i K\partial_i F^{-1} , \quad (5.12)$$

or, combined with (5.10),

$$\partial^2 F^{-1} = 2b^i \partial_i F^{-1} , \quad \partial^2 K = 2b^i \partial_i K , \quad \phi = \phi_0 + b_i x^i + \frac{1}{2} \ln F . \quad (5.13)$$

### 5.3. General chiral null model

For the general chiral null model (with  $u$  dependence), one can set  $K = 0$  by the gauge transformation (2.8). Adding sources and integrating over  $v$  and  $u$  as above we arrive at the following effective  $x$ -theory

$$\begin{aligned} L_{eff} = & \partial x_i \bar{\partial} x^i - F^{-1}(x) \partial U \bar{\partial} V + 2A_i(x, \partial^{-1}[F^{-1}(x) \partial U]) \partial U \bar{\partial} x^i \\ & + \alpha' \mathcal{R} \phi'(x, \partial^{-1}[F^{-1}(x) \partial U]) + X_i \partial \bar{\partial} x^i , \end{aligned} \quad (5.14)$$

where  $\phi'$  is as in (5.8) and we again use a special scheme to keep the free kinetic term of  $x^i$  unchanged (see [1]). The condition of conformal invariance in the  $\partial x \bar{\partial} x$  direction is straightforward generalization of (5.3) and the condition in the model with  $A_i = 0$  (5.9), i.e.  $\phi' = \phi(u) + b_i x^i$ . The  $\partial U \bar{\partial} V$  term is conformally invariant, provided one imposes (5.10) as in the  $A_i = 0$  model. The conditions of conformal invariance in the  $\partial u \bar{\partial} u$  and  $\partial u \bar{\partial} x$  directions are similar to (5.4) with  $K = 0$ ,

$$\partial^i \partial_u A_i - 2b^i \partial_u A_i + 2F^{-1} \partial_u^2 \phi = 0 , \quad -\frac{1}{2} \partial_i \mathcal{F}^{ij} + b_i \mathcal{F}^{ij} = 0 . \quad (5.15)$$

The reason why there are no extra terms involving  $F$  is that the locality of the conformal anomaly implies that the only contributions depending on derivatives of  $F$  are tadpole ones which thus vanish due to (5.10). This is easy to see by expanding the argument  $x^i(z)$  of  $F^{-1}$  and  $A_i$  near its ‘classical’ value. Contractions of the quantum fields on the opposite sides of the  $\partial^{-1}$ -operator produce only non-local contributions to the corresponding effective action.

Equation (5.15) is valid in the gauge  $K = 0$ . The general form of this conformal invariance condition can be obtained by doing the gauge transformation (2.8). Combining all the conditions together we obtain<sup>12</sup>

$$-\frac{1}{2} \partial^2 F^{-1} + b^i \partial_i F^{-1} = 0 , \quad \phi = \phi(u) + b_i x^i + \frac{1}{2} \ln F(x) , \quad (5.16)$$

$$-\frac{1}{2} \partial^2 K + b^i \partial_i K + \partial^i \partial_u A_i - 2b^i \partial_u A_i + 2F^{-1} \partial_u^2 \phi = 0 , \quad -\frac{1}{2} \partial_i \mathcal{F}^{ij} + b_i \mathcal{F}^{ij} = 0 . \quad (5.17)$$

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<sup>12</sup> Let us note that the fact that the model (2.7) is Weyl invariant means also that when considered on a flat world sheet this  $\sigma$ -model is ultra-violet finite to all loop orders *on the mass shell*. The latter clarification means that the standard  $\beta$ -functions vanish only modulo a diffeomorphism term (which is related to the presence of a non-trivial dilaton in the corresponding Weyl-invariant model).

#### 5.4. Further generalizations?

Can one extend the chiral null model (2.6) to include a larger class of backgrounds and maintain their conformal invariance? As we have already remarked, one possible generalization is to replace the transverse space with a nontrivial conformal field theory. Another possibility would appear to be the addition of a second vector coupling

$$L = F(x)\partial u\bar{\partial}v + \tilde{K}(x, u)\partial u\bar{\partial}u + 2\tilde{A}_i(x, u)\partial u\bar{\partial}x^i \quad (5.18)$$

$$+ 2\tilde{S}_i(x, u)\partial x^i\bar{\partial}v + \partial x_i\bar{\partial}x^i + \alpha'\mathcal{R}\phi(x, u) .$$

This  $\sigma$ -model shares with the chiral null model the following three properties:

- (i) conformal invariance of the transverse part of the model;
- (ii) existence of an affine symmetry  $v' = v + h(\tau + \sigma)$  in a null direction;
- (iii) chirality of all vector couplings.

The second condition implies the existence of the associated conserved chiral current. At the ‘point-particle’ (zero mode) level this affine stringy symmetry reduces to the null Killing symmetry  $v' = v + h$ ,  $h = \text{const}$ .

However, the model (5.18) is not, in general, conformal to all orders if only the leading-order equations are satisfied. As before, we can still explicitly integrate out  $v$  and then  $u$ . But the result is a complicated  $x$ -theory for which the conditions of conformal invariance seem difficult to formulate and solve explicitly to all orders.<sup>13</sup>

To illustrate this point, let us consider a particular example of (5.18) with  $F = 1$ ,  $\tilde{A}_i = 0$  and  $u$ -independent couplings,

$$L = \partial u\bar{\partial}v + \tilde{K}(x)\partial u\bar{\partial}u + 2\tilde{S}_i(x)\partial x^i\bar{\partial}v + \partial x_i\bar{\partial}x^i + \alpha'\mathcal{R}\phi(x) . \quad (5.19)$$

The corresponding target space metric has a null Killing vector, but in contrast to the case of the model (2.6) with  $F = 1$  this vector is not covariantly constant. Making the coordinate transformation  $u \rightarrow u + p(x)$  we get

$$L = \partial u\bar{\partial}v + \tilde{K}\partial u\bar{\partial}u + \tilde{K}\partial_i p(\partial u\bar{\partial}x^i + \partial x^i\bar{\partial}u) \quad (5.20)$$

$$+ (2\tilde{S}_i + \partial_i p)\partial x^i\bar{\partial}v + (\delta_{ij} + \tilde{K}\partial_i p\partial_j p)\partial x^i\bar{\partial}x^j + \alpha'\mathcal{R}\phi(x) .$$

If we now choose  $\tilde{S}_i = -\frac{1}{2}\partial_i p$ , the new  $\partial x\bar{\partial}v$ -coupling disappears. We learn that in this case the model (5.20) is equivalent to a modification of (2.6) with a non-trivial transverse metric and non-chiral  $\partial u\bar{\partial}x$  and  $\partial x\bar{\partial}u$  - couplings (cf. (2.14)). Integrating over  $v$  it is easy to see that the the resulting conformal invariance conditions (both in  $\partial u\bar{\partial}u$  and  $\partial x\bar{\partial}x$  directions) contain non-trivial corrections to all orders in  $\alpha'$ .

This example makes it clear that the above three conditions are not sufficient to ensure that leading order solutions will be exact. One needs an additional condition which can be taken to be

- (iv) the null Killing vector should be orthogonal to the transverse subspace.

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<sup>13</sup> We assume that  $\tilde{K}$  or  $\tilde{A}_i$  do not vanish at the same time. In the special case when  $\tilde{K} = 0$  and  $\tilde{A}_i = 0$  the model (5.18) is equivalent to the special case (2.18) of (2.6) with  $u \rightarrow v$ ,  $v \rightarrow u$ .

One can further generalize (5.18) by introducing a non-trivial transverse space metric. Then there may exist some special cases in which such a model may remain conformal to all orders once it is conformal to the leading order. An example is provided by

$$L = F(x) [\partial u + 2S_i(x)\partial x^i] [\bar{\partial} v + 2A_i(x)\bar{\partial} x^i] + \partial x_i \bar{\partial} x^i + \alpha' \mathcal{R}\phi(x) . \quad (5.21)$$

This model is related by  $u$ -duality to the  $u$ -independent case of the ‘non-chiral’ generalization of the  $K$ -model (2.14) with two non-vanishing vector couplings (the relation of the functions is  $F = K^{-1}(x)$ ,  $S_i = \bar{A}_i(x)$ ,  $A_i = A_i(x)$ ,  $\phi = \phi_0 + \frac{1}{2}\ln F(x)$ ). In the case when  $S_i$  and  $A_i$  have constant field strengths (2.17), the theory (5.21), like (2.14), can be shown to be conformally invariant to all loop orders, provided (cf. (2.16))

$$-\frac{1}{2}\partial^2 F^{-1} + \mathcal{F}^{ij}\bar{\mathcal{F}}_{ij} = 0 , \quad \phi = \phi_0 + \frac{1}{2}\ln F , \quad \bar{\mathcal{F}}_{ij} = 2\partial_{[i}S_{j]} . \quad (5.22)$$

The proof is a simple version of the arguments used in the previous subsections (in the special case of  $S_i = -A_i$  it was given already in the Appendix B of [1]). Introducing the sources and integrating out  $u$  and  $v$  one obtains the following effective  $x$ -theory (cf. (5.2), (5.7), (5.8))

$$\begin{aligned} L_{eff} = & \partial x_i \bar{\partial} x^i - F^{-1}(x)\partial U \bar{\partial} V + 2A_i(x)\partial U \bar{\partial} x^i + 2S_i(x)\partial x^i \bar{\partial} V \\ & + \alpha' \mathcal{R}\phi'(x) + X_i \partial \bar{\partial} x^i , \end{aligned} \quad (5.23)$$

so that if  $A_i$  and  $S_i$  are linear in  $x$  all conformal anomaly contributions come only from one-loop diagrams.

## 6. Superstring and heterotic string solutions

So far we have discussed exact classical solutions of the bosonic theory. A generalization to the case of the closed superstring theory is straightforward. The superstring action is given by the (1,1) supersymmetric extension of the bosonic  $\sigma$ -model (2.6) (with  $x^\mu = (u, v, x^i)$  in (2.6) replaced by (1,1) superfields  $\hat{X}^\mu(z, \theta, \bar{\theta})$ ). Repeating the arguments of section 5 starting with the (1,1) supersymmetric extension of (2.6)  $I_{(1,1)} = \int d^2z d^2\theta (G_{\mu\nu} + B_{\mu\nu})(\hat{X}) \mathcal{D}\hat{X}^\mu \bar{\mathcal{D}}\hat{X}^\nu$  and using that the one-loop conformal invariance conditions are the same as in the bosonic case one finds that our exact bosonic backgrounds also represent superstring solutions. One can also start with the component representation (here  $\hat{\omega}_{\pm n\mu}^m = \omega_{n\mu}^m \pm \frac{1}{2}H_{n\mu}^m$ )

$$\begin{aligned} I_{(1,1)} = & \int d^2z [(G_{\mu\nu} + B_{\mu\nu})(x) \partial x^\mu \bar{\partial} x^\nu + \lambda_{Rm}(\delta_n^m \bar{\partial} + \hat{\omega}_{-n\mu}^m(x) \bar{\partial} x^\mu) \lambda_R^n \\ & + \lambda_{Lm}(\delta_n^m \partial + \hat{\omega}_{+n\mu}^m(x) \partial x^\mu) \lambda_L^n - \frac{1}{2} \hat{R}_{+mnpq} \lambda_L^m \lambda_L^n \lambda_R^p \lambda_R^q] , \end{aligned} \quad (6.1)$$

write down the fermionic part of the action explicitly with the help of (A.9), (A.16) and directly integrate over the left and right fermions. One then finds that the only effect of the fermionic contributions on the effective bosonic  $x^i$ -theory is to cancel the local  $\partial \ln F \bar{\partial} \ln F$

term coming from the bosonic  $u, v$ -determinant.<sup>14</sup> Thus there is no need for a special adjustment of a scheme compared to the pure bosonic case (see also [1]).

As for the heterotic string solutions, one approach is to start with a closed superstring solution and embed it into a heterotic string theory by identifying the generalized Lorentz connection  $\hat{\omega}_{+n\mu}^m$  (or  $\hat{\omega}_{-n\mu}^m$ ) with a Yang-Mills background, i.e. by rewriting the  $(1, 1)$  supersymmetric  $\sigma$ -model in the  $(1, 0)$  (or  $(0, 1)$ ) supersymmetric heterotic  $\sigma$ -model form [33, 34, 35, 36]. For this to be possible, the holonomy group of the generalized connection  $\hat{\omega}_+$  (or  $\hat{\omega}_-$ ) should be a subgroup of the heterotic string gauge group. In general, such embedding is problematic for solutions with a curved space-time (i.e. with a non-trivial time-like direction) since the holonomy is then (a subgroup of) a non-compact Lorentz group  $SO(1, D-1)$  while the heterotic gauge group should be compact on unitarity grounds.<sup>15</sup> In fact, as shown in Appendix A.2, the holonomy groups of  $\hat{\omega}_+$  and  $\hat{\omega}_-$  for generic chiral null models are non-compact (except for the case of the plane wave background (4.10) when the holonomy of  $\hat{\omega}_+$  is  $SO(D-2)$ ) and thus cannot be embedded into  $SO(32)$  or  $E_8 \times E_8$ .

### 6.1. Exact heterotic string solutions

One should thus try a more direct approach. As indicated above, given a bosonic string theory, there exist, in principle, two possible ways to construct a heterotic string theory depending on whether the “right” or “left” parts of the bosonic coordinates are supersymmetrized, i.e. on whether one considers a  $(1, 0)$  or  $(0, 1)$  supersymmetric world sheet theory. The two heterotic theories are related by interchanging left- and right- movers in the vertex operators, and, in general, are inequivalent. The fermionic parts of the heterotic  $\sigma$ -models corresponding to the two theories depend on  $\hat{\omega}_-$  and  $\hat{\omega}_+$  respectively.<sup>16</sup> In what follows we shall concentrate on the standard  $(1, 0)$  (or “right”) theory since it turns out that the  $(0, 1)$  (or “left”) theory does not have chiral null models as exact solutions.

The action of the  $(1, 0)$  heterotic  $\sigma$ -model is given by (we ignore the “internal”

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<sup>14</sup> A simple test that this cancellation does take place is provided by the observation that the two-loop  $\beta$ -function must vanish (in a “supersymmetric” scheme) in the  $(1, 1)$  supersymmetric  $\sigma$ -model [32], while the one-loop induced term  $\partial \ln F \bar{\partial} \ln F$  term would contribute to the two-loop conformal anomaly.

<sup>15</sup> A special case of this was pointed out in [37]. Notice that if the gauge group is non-compact, at least one of the internal fermions has a negative norm but (compared to the  $(1, 1)$  supersymmetric superstring case) there is no extra local world sheet superconformal symmetry to gauge it away [38].

<sup>16</sup> In particular, the  $\sigma$ -model  $\beta$ -functions and low-energy effective actions corresponding to the two theories are related by simply changing the sign of  $B_{\mu\nu}$  (the effective actions of bosonic or supersymmetric string theories are invariant under  $B_{\mu\nu} \rightarrow -B_{\mu\nu}$  since these theories are invariant under world sheet parity transformation). That implies, e.g., that the “right” and “left” heterotic extensions of a bosonic background which is chiral (i.e. which distinguishes between left and right, e.g., having  $B_{\mu\nu} \neq 0$ ) will be inequivalent.

fermionic part with a possible gauge field background)

$$\begin{aligned}
I_{(1,0)} &= \int d^2z d\theta (G_{\mu\nu} + B_{\mu\nu})(\hat{X}) \mathcal{D}\hat{X}^\mu \bar{\partial}\hat{X}^\nu \\
&= \int d^2z [(G_{\mu\nu} + B_{\mu\nu})(x) \partial x^\mu \bar{\partial} x^\nu + \lambda_{Rm}(\delta_n^m \bar{\partial} + \hat{\omega}_{-n\mu}^m(x) \bar{\partial} x^\mu) \lambda_R^n] .
\end{aligned} \tag{6.2}$$

The (1,1) superstring  $\sigma$ -model action (6.1) can be formally obtained from (6.2) by adding the internal left fermionic part coupled to the gauge field background equal to  $\hat{\omega}_+$ .

Thus  $\hat{\omega}_-$  appears in the fermionic part of the  $\sigma$ -model action (6.2) (and also in the leading-order space-time supersymmetry transformation laws). The  $\beta$ -functions and the effective action  $S$  of this theory will depend on  $\hat{\omega}_-$  *but also* explicitly on the curvature  $R$  of  $G_{\mu\nu}$  and the antisymmetric tensor field strength  $H$ . The  $\sigma$ -model anomaly will also naturally involve  $\hat{\omega}_-$ . However, since the form of the anomaly is ambiguous (scheme dependent) [39,40] it can be arranged so that it will be  $\hat{\omega}_+$  that will enter the anomaly relation as well as the “anomaly-related” terms in the effective action (this, in fact, is a common assumption, see e.g. [41,42]). It should be emphasized that there is *no* unambiguous definition of such “anomaly-related” terms since  $S$  is scheme dependent and, in general, cannot be represented only in terms of  $\hat{\omega}_+$ . There are always other  $H$ -dependent terms which are not expressed in terms of the generalized curvature of  $\hat{\omega}_+$  (so that one can equally well use  $\hat{\omega}_-$  in place of  $\hat{\omega}_+$  at the expense of modifying the rest of the terms).<sup>17</sup>

Let us now show that our bosonic solutions are exact solutions of the heterotic string theory without any extra gauge-field background: the direct (1,0) supersymmetric extension of the bosonic  $\sigma$ -model (2.6) is conformally invariant if the bosonic model is conformal. The fermionic part of the action (6.2) does not actually contribute to the conformal anomaly. This follows from the special “null” holonomy property of  $\hat{\omega}_-$ : according to Appendix A (see (A.16)) the only non-vanishing component of the generalized Lorentz connection  $\hat{\omega}_-$  is  $\hat{\omega}_{-\hat{u}\hat{i}\mu}$  ( $\hat{u}, \hat{v}, \hat{i}$  are tangent space indices).<sup>18</sup> The non-trivial fermionic terms in (6.2)

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<sup>17</sup>  $O(\alpha')$ -terms in the heterotic string effective action were computed in [43] and [21] starting from the string  $S$ -matrix. As was shown in [21], there exists a scheme in which the  $\alpha'$ -term (its part which is not related to Chern-Simons modification of the leading-order  $H^2$ -term) in the heterotic string action is the same as in the bosonic string one up to an extra overall factor of 1/2. The same result was obtained from the analysis of the 3-loop conformal anomaly of the heterotic  $\sigma$ -model [44].

<sup>18</sup> This property of  $\hat{\omega}_-$  is also responsible for the “one-half” extended space-time supersymmetry of our bosonic backgrounds when they are embedded into  $D = 10$  supergravity as shown for the special cases of the (generalized) FS in [6,9] and for the  $F = 1$  and  $K = F^{-1}$  cases in [22,23] (our notation for  $\hat{\omega}_-$  and  $\hat{\omega}_+$  are opposite to that of [22,23]). The general chiral null model also has unbroken spacetime supersymmetry, at least to leading order in  $\alpha'$ . It should be possible to address higher order corrections to the spacetime supersymmetry transformations for this model in the worldsheet approach using Green-Schwarz superstring action in a light-cone type gauge (cf. [9]).

are thus given by

$$L_{(1,0)}(\lambda_R) = \lambda_R^{\hat{u}} \bar{\partial} \lambda_R^{\hat{v}} + \lambda_R^{\hat{i}} \bar{\partial} \lambda_R^{\hat{j}} + \hat{\omega}_{-\hat{u}\hat{i}\mu}(x) \bar{\partial} x^\mu \lambda_R^{\hat{u}} \lambda_R^{\hat{i}} . \quad (6.3)$$

The “null” structure of the coupling implies that integrating out fermions does not produce a non-trivial contribution to the  $x^\mu$ -theory which remains conformally invariant. There is an obvious similarity with integrating out  $u$  and  $v$  in the bosonic theory (cf. Section 5).

Thus we do not need a non-trivial gauge field background to promote our bosonic solutions to heterotic ones. We conclude that, for example, the exact  $D = 5$  bosonic solutions (4.4) are also heterotic string solutions and so are their four dimensional ‘images’ (4.6). In particular, the  $D = 4$  extremal electric black holes discussed in Section 4.2 are thus exact heterotic string solutions [2] without any extra gauge field background.

Let us compare the above conclusion with the perturbative result for the two-loop  $\beta$ -function of the heterotic  $\sigma$ -model. Let us consider the “non-anomalous”  $\alpha'$  contribution to the metric  $\beta$ -function  $\beta_{\mu\nu}^G$  (i.e. we shall ignore other non-covariant  $\alpha'$ -corrections which modify the one-loop  $H^2$ -term by the Chern-Simons terms). The contribution of the fermions  $\lambda_R$  is essentially the same form as the standard two-loop “ $F^2$ ”-term that comes from the internal fermionic sector  $\psi_L$  [34] except for the fact that the gauge field is represented by the connection  $\hat{\omega}_-$  [45]. Thus

$$(\beta_{\mu\nu}^{G(2)})_{(1,0)} = (\beta_{\mu\nu}^{G(2)})_0 - \frac{1}{4} \alpha' \hat{R}_{-mn\lambda\mu} \hat{R}_{-}^{mn\lambda}{}_\nu , \quad (6.4)$$

where  $(\beta_{\mu\nu}^{G(2)})_0$  is the bosonic contribution. There exists a special chiral “right” scheme in which the latter is given by [21]

$$(\beta_{\mu\nu}^{G(2)})_0 = \frac{1}{4} \alpha' \left( 2 \hat{R}_{-}^{\alpha\beta\lambda}{}_{(\mu} \hat{R}_{-\nu)\alpha\beta\lambda} - \hat{R}_{-}^{\beta\lambda\alpha}{}_{(\mu} \hat{R}_{-\nu)\alpha\beta\lambda} + \hat{R}_{-\alpha(\mu\nu)\beta} H^{\alpha\rho\sigma} H^{\beta}{}_{\rho\sigma} \right) . \quad (6.5)$$

As follows from (A.9)  $\hat{R}_{-mn\lambda\mu} \hat{R}_{-}^{mn\lambda}{}_\nu$  (i.e. the fermionic contribution) indeed vanishes for our backgrounds. As for (6.5), it also vanishes when  $F = 1$  but in general one needs to choose a different scheme to avoid  $\alpha'$ -corrections (see [1]).

Given the scheme dependence of the  $\beta$ -function, in the *heterotic*  $\sigma$ -model context there may exist a scheme in which the bosonic contribution to the  $\sigma$ -model  $\beta$ -function (6.4) can be put in the following “left-right symmetric” form

$$(\beta_{\mu\nu}^{G(2)})_0 = \frac{1}{4} \alpha' \left( \hat{R}_{+mn\lambda\mu} \hat{R}_{+}^{mn\lambda}{}_\nu + \hat{R}_{-mn\lambda\mu} \hat{R}_{-}^{mn\lambda}{}_\nu \right) . \quad (6.6)$$

Including the gauge field contribution of the internal left fermions the heterotic  $\sigma$ -model  $\beta$ -function corresponding to this “symmetric” scheme then is given by

$$(\beta_{\mu\nu}^{G(2)})_{(1,0)} = \frac{1}{4} \alpha' \hat{R}_{+mn\lambda\mu} \hat{R}_{+}^{mn\lambda}{}_\nu - \frac{1}{4} \alpha' F_{IJ\lambda\mu}(V) F^{IJ\lambda}{}_\nu(V) . \quad (6.7)$$

This expression is consistent with the expectation that the two-loop  $\beta$ -function should vanish once we identify the gauge field  $V_\mu^{IJ}$  with  $\hat{\omega}_+$  since then the heterotic  $\sigma$ -model becomes identical to the (1,1) supersymmetric model (6.1). The two-loop contribution

(6.7) with  $V_\mu = 0$  does not vanish for our backgrounds even in the simplest plane-wave case  $F = 1$ . As already mentioned above, in general, we cannot make it vanish by the identification  $V = \hat{\omega}_+$  since the holonomy group of  $\hat{\omega}_+$  is non-compact. Thus in this scheme our solutions will be modified by higher-order  $\alpha'$  corrections.

In the special case of  $F = 1, K = 0, A_i = A_i(x)$ , the only non-vanishing component of  $\hat{\omega}_+$  is  $\hat{\omega}_{+\hat{i}\hat{j}u} = -\mathcal{F}_{ij}$  and one finds that  $\hat{R}_{+mn\lambda\mu}\hat{R}_+^{mn\lambda}{}_\nu$  in (6.7) has non-vanishing  $uu$ -component equal to  $(\partial_k \mathcal{F}_{ij})^2$ . If we set  $V_\mu = 0$  and start with the leading-order solution  $K = 0, \partial_i \mathcal{F}^{ij} = 0$  then  $K$  receives the  $\alpha'$ -correction  $K_1$  satisfying (cf. (A.27))  $-\frac{1}{2}\partial^i \partial_i K_1 + \frac{1}{4}\alpha'(\partial_k \mathcal{F}_{ij})^2 = 0$ , i.e.  $K_1 = \frac{1}{4}\alpha'(\mathcal{F}_{ij})^2$ . Such modification can be thought of as a local field redefinition corresponding to the transformation from the chiral “right” scheme (6.4),(6.5) where  $K_1 = 0$  to the “left-right symmetric” scheme (6.7).<sup>19</sup>

Since in this exceptional case the holonomy of  $\hat{\omega}_+$  is compact ( $SO(D-2)$ ), there is also an alternative option to introduce the gauge field background  $V_\mu^{IJ}$  equal to  $\hat{\omega}_+$  and in this way cancel the higher order correction. This was suggested in [22] where (6.7) was assumed to be the form of the  $\alpha'$ -correction in the heterotic string equation of motion.<sup>20</sup> As we have mentioned, the idea of embedding of  $\hat{\omega}_+$  into the gauge group does not have consistent generalizations to other cases except the one with  $F = 1, K = 0, A_i = A_i(x)$  so that we disagree with the suggestion of [22,23] that  $F = 1$  and  $F = K^{-1}$  models are exact heterotic string solutions only when supplemented by a gauge field background. The need to introduce a non-trivial gauge field background in [30,23] was caused by having implicitly taken the  $\alpha'$  term in the effective action in a specific “symmetric scheme” (in which  $\hat{\omega}_+$  appears in the “anomaly-related” terms). As we have explained above, the form of  $\alpha'$ -corrections is scheme dependent and in the natural chiral scheme there is no need for an extra gauge field background at all.

The plane wave model (4.10) with  $F = 1, K = 0, A_i = A_i(x)$  and the gauge field background  $V_u^{ij} = \hat{\omega}_{+\hat{i}\hat{j}u} = -\mathcal{F}_{ij}$  is equivalent to the (1,1) supersymmetric superstring  $\sigma$ -model and thus represents an exact solution according to the discussion at the beginning of this section. It is instructive to see explicitly why the resulting model remains conformally invariant: the fermionic terms now are ( $\psi_L^m$  are internal fermions; see also (A.9),(A.16))

$$\begin{aligned} L_{(1,1)}(\lambda_R, \psi_L) &= \lambda_R^{\hat{u}} \bar{\partial} \lambda_R^{\hat{v}} + \lambda_R^{\hat{i}} \bar{\partial} \lambda_R^{\hat{j}} + \mathcal{F}_{ij}(x) \bar{\partial} x^j \lambda_R^{\hat{u}} \lambda_R^{\hat{i}} \\ &+ \psi_L^{\hat{u}} \partial \psi_L^{\hat{v}} + \psi_L^{\hat{i}} \bar{\partial} \psi_L^{\hat{j}} - \mathcal{F}_{ij}(x) \partial u \psi_L^{\hat{i}} \psi_L^{\hat{j}} - \frac{1}{2} \partial_i \mathcal{F}_{jk}(x) \lambda_R^{\hat{u}} \lambda_R^{\hat{i}} \psi_L^{\hat{j}} \psi_L^{\hat{k}}. \end{aligned} \quad (6.8)$$

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<sup>19</sup> This redefinition of  $G_{uu}$  could be thought of as induced by  $G'_{\mu\nu} = G_{\mu\nu} + \frac{1}{4}\alpha' H_{\mu\lambda\rho} H_\nu^{\lambda\rho}$ . It may also be related to the non-covariant redefinition  $G'_{\mu\nu} = G_{\mu\nu} + \frac{1}{4}\alpha'(\hat{\omega}_+^{mn}{}_\mu \hat{\omega}_{+mn\nu} - V_\mu^{IJ} V_{IJ\nu})$  used in [40] in order to preserve world sheet supersymmetry (there is only  $\hat{\omega}_+ \hat{\omega}_+$ -term if  $V = 0$  and the whole redefinition is trivial if  $V = \hat{\omega}_+$ ).

<sup>20</sup> While the effective action considerations in [22,23] are not sufficient to demonstrate the exactness of the solutions to all orders in  $\alpha'$  since they were ignoring “anomaly-unrelated” terms (in particular, no explanation was given why these backgrounds are superstring solutions, i.e. why the corresponding (1,1) supersymmetric  $\sigma$ -model should be conformally invariant), this is possible within our direct world sheet approach. Although the approach of [22,23,37] is incomplete, our present work was much motivated and influenced by these interesting papers.

Integration over  $\lambda_R^{\hat{v}}$  ‘freezes’ out  $\lambda_R^{\hat{u}}$ , while the term  $\mathcal{F}_{ij}(x)\partial u\psi_L^{\hat{i}}\psi_L^{\hat{j}}$  does not produce new divergencies in the  $uu$ -direction since the total action does not contain local  $\bar{\partial}u$ -couplings (cf. (5.11)).

Finally, let us consider the (0,1) (“left”) heterotic theory. Here the superstring fermions are coupled to  $\hat{\omega}_+$ . Since according to (A.17)  $\hat{\omega}_+$  has general holonomy, one should expect non-trivial fermionic contributions to the conformal anomaly. The gauge field background cannot be consistently introduced since the (abelian) holonomy group of  $\hat{\omega}_-$  is “null” (non-compact). The corresponding leading-order solutions thus should have corrections to all orders in  $\alpha'$ . Given that  $\hat{\omega}_+$  (which in this theory appears also in the space-time supersymmetry transformation laws) is of generic form, one should not also expect to find Killing spinors, i.e. a residual supersymmetry.

## 6.2. Extended world sheet supersymmetry

It is clear that the abelian gauge fields of the four dimensional solutions (4.6) have a Kaluza-Klein and not a heterotic Yang-Mills origin. In general, given a  $D = 4$  leading-order bosonic background, its embedding into the heterotic string theory is not unique. The embeddings of extremal  $D = 4$  dilatonic black holes in which the  $U(1)$  gauge fields have a Kaluza-Klein (i.e.  $N = 4$  supergravity) and not a heterotic Yang-Mills origin have extended (e.g.  $N = 2$ ,  $D = 4$ ) space-time supersymmetry [46]. Since our general bosonic  $D \leq 10$  backgrounds have extended space-time supersymmetry when embedded into  $D = 10$  supergravity theory [6,9,22,23,13] one may also try to invoke supersymmetry to argue that they are exact superstring solutions.

In fact, the presence of extended space-time supersymmetry suggests (cf. [47,42]) that the corresponding (1,0) supersymmetric  $\sigma$ -models may have extended world sheet supersymmetry. If the latter supersymmetry is large enough, one may use the fact that there exists a scheme in which the  $(4,n)$  supersymmetric  $\sigma$ -models are conformal to all orders [48].

In contrast to our approach described in Section 5 and in the previous subsection, any argument based on extended world sheet supersymmetry is bound to have a limited applicability. The standard discussions of extended world sheet supersymmetry apply to the case of Euclidean target space signature. To have  $(2,n)$  supersymmetry the dimension  $D$  must be even; the  $(4,n)$  supersymmetry is possible only when  $D$  is multiple of 4. Most of our models (e.g. all with odd spacetime dimension) do not admit extended world sheet supersymmetry since they do not admit a complex structure when analytically continued to Euclidean signature.

The generic chiral null model (2.6) does not have a natural analytic continuation with a real Euclidean target space metric. For example, if one analytically continues  $u + v$  keeping  $u - v$  real, so that  $u$  and  $v$  become complex conjugates ( $v = \bar{u}$ ), then the metric is no longer real unless  $K$  and  $A_i$  in (2.6) are taken to be zero. There may exist a well-defined Euclidean analog of (2.6) for some special choice of  $A_i$  but we shall ignore this possibility for simplicity. In the special case of the  $F$ -model (2.3) one gets a real action on the Euclidean world sheet (but thus a complex string action in the Minkowski world sheet signature case)

$$L = F(x)\partial u\bar{\partial}\bar{u} + \partial x_i\bar{\partial}x^i + \alpha'\mathcal{R}\phi(x) . \quad (6.9)$$

The corresponding Euclidean metric  $ds^2 = F(x)dud\bar{u} + dx_idx^i$  is real but the antisymmetric tensor is imaginary. If the dimension is even,  $D = 2N$ , the metric is hermitian.



Replacing  $x^i$  by a set of complex coordinates  $w_s$  ( $s = 1, \dots, N-1$ ) the metric and the antisymmetric tensor are

$$ds^2 = F(w, \bar{w}) du d\bar{u} + dw_s d\bar{w}_s, \quad B_{u\bar{u}} = \frac{1}{2}F, \quad H_{su\bar{u}} = \frac{1}{2}\partial_s F. \quad (6.10)$$

The corresponding  $(1,0)$  supersymmetric  $\sigma$ -model admits  $(2,0)$  extended supersymmetry. This is clear from the comparison with the conditions on geometry implied by  $(2,0)$  supersymmetry [36], as reviewed, e.g., in [49] (for some earlier discussions of related complex geometries, see [50,51]). Provided  $\partial^2 F^{-1} = 0$  the generalized connection with torsion has special (not  $U(N)$  but  $SU(N)$ ) holonomy and satisfies the generalized quasi Ricci flatness condition (see Appendix A)

$$\hat{R}_{\mu\nu} = \hat{D}_\mu V_\nu, \quad V_\mu = -\partial_\mu \ln F. \quad (6.11)$$

If the dimension  $D$  is a multiple of four, i.e.  $N = 2N'$ , a  $(2,0)$  supersymmetric  $\sigma$ -model may admit  $(4,0)$  extended supersymmetry. In fact, the Euclidean  $F$ -model (6.9) does have it, as is clear again from the comparison with the expressions in [49]. In particular, the holonomy of the generalized connection is an  $Sp(N')$  subgroup of  $SU(N)$ .<sup>21</sup> Given that  $(4,0)$  supersymmetric  $\sigma$ -models are conformally invariant to all orders (in a properly chosen scheme) [48] we get (for  $D = 4N'$ ) an independent proof of the fact that the  $F$ -model corresponds to an exact solution of heterotic string theory.<sup>22</sup>

It should be stressed again that our explicit proof given in [1] and in the present paper is more direct and applies for any  $D$  as well as to a more general class of models (2.6). In general, a relevant property which is important for the proof of exactness is the special holonomy of the generalized connection with torsion and not an extended supersymmetry (which is just a consequence of the special holonomy under certain additional conditions like existence of a complex structure).<sup>23</sup>

### 6.3. Relation to other $D = 4$ heterotic solutions

What about non-supersymmetric solutions of  $D = 4$  heterotic string theory? For example, the charged dilatonic black hole may be considered as a non-supersymmetric leading-order solution [12] of the  $D = 4$  heterotic string theory with the charge being that

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<sup>21</sup> In the simplest case of  $D = 4$  and  $F = F(|w|)$  the metric becomes conformal to a Kähler metric, cf. [42,49].

<sup>22</sup> The conclusion about extended supersymmetry of the  $F$ -models is consistent with the fact that some of them correspond to special nilpotently gauged WZW models [5]. The latter are formulated essentially in terms of the WZW model on a (maximally non-compact) group  $G$  and thus their Euclidean versions should admit  $(2,0)$  or  $(2,2)$  supersymmetry [51,52].

<sup>23</sup> A somewhat related remark was made in [53], where it was pointed out that since the  $\sigma$ -model on a Calabi-Yau space has a special holonomy it thus has an extra infinite-dimensional non-linear classical symmetry. That symmetry (if it were not anomalous at the quantum level) would rule out all higher-loop corrections to the  $\beta$ -function [53]. In our case, the analogous symmetry is linear and is the affine symmetry generated by the null chiral current.

of the  $U(1)$  subgroup of the Yang-Mills gauge group. This solution must have an extension to higher orders in  $\alpha'$  which, in general, may not be the same as the above supersymmetric ‘Kaluza-Klein’ solution obtained by dimensional reduction from  $D = 5$ . Even though the leading-order terms in the compactified (from  $D = 5$  to  $D = 4$ ) bosonic string theory and  $D = 4$  heterotic string theory with a  $U(1)$  gauge field background look the same, the  $\alpha'$ -corrections are different, so that our bosonic result does not automatically imply that the extremal electric black hole considered as a  $D = 4$  heterotic string solution is also exact. In fact, it is known that the non-supersymmetric extremal magnetic solution of the  $D = 4$  heterotic string has  $\alpha'$ -corrections [54]. The same is likely to be true for the non-supersymmetric extremal electric solution.

To explain this difference between “supersymmetric” and “nonsupersymmetric” solutions it is useful to consider the space-time effective action approach. Our exact  $D = 4$  solutions (4.7) obtained by dimensional reduction are actually  $D = 5$  bosonic or heterotic string solutions. This means that there exists a choice of (five dimensional) field redefinitions in which the  $D = 5$  effective equations evaluated on our background contain no  $\alpha'$  corrections. As shown in section 3, the dimensional reduction of the  $D = 5$  action includes *two* gauge fields (as well as an extra scalar modulus field). Even though these two gauge fields are equal for our solution (4.7), the general field redefinition treats them independently. In contrast, the  $D = 4$  non-supersymmetric heterotic action contains a single gauge field and thus a smaller group of field redefinitions. Thus the fact that nontrivial  $\alpha'$  corrections inevitably arise in this case (for the magnetically charged black hole [54] and, most likely, for the electrically charged case as well) does not contradict our claim that the supersymmetric electrically charged solution obtained from dimensional reduction is exact.

In general, the starting point is the  $D = 10$  heterotic string with the leading-order term in the effective action being represented by the  $N = 1$ ,  $D = 10$  supergravity coupled to  $D = 10$  super Yang-Mills theory. Compactified on a 6-torus, this effective action becomes that of  $N = 4$ ,  $D = 4$  supergravity coupled to a number of abelian  $N = 4$  vector multiplets and  $N = 4$  super Yang-Mills. The simplest charged dilatonic black hole solution may be embedded in this theory in several inequivalent ways, depending on which vector field(s) is kept non-vanishing. The dependence of higher-order  $\alpha'$ -terms in the full effective action on different vector fields is different, so it should not be surprising that the solutions that happen to coincide at the leading-order level may turn out to receive different  $\alpha'$ -corrections.

Finally, let us note that it may be possible to utilize some of our  $D > 4$  exact bosonic solutions to construct other  $D = 4$  heterotic string solutions.<sup>24</sup> The idea is to start with an exact higher dimensional bosonic solution and then fermionize the ‘internal’ coordinates in an appropriate way to obtain a heterotic  $\sigma$ -model. A similar method was used in [55] to find the heterotic solution representing a  $D = 2$  monopole theory (which was related to the throat limit of the  $D = 4$  extreme magnetically charged black hole) and in [56] to describe a non-trivial throat limit of the  $D = 4$  dilatonic Taub-NUT solution [13,14].

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<sup>24</sup> To establish a relation between heterotic and bosonic models one can use the following strategy: start with a leading-order heterotic string solution, write down the corresponding heterotic  $\sigma$ -model and then try to bosonize it to put it in a form of a bosonic  $\sigma$ -model for which it may be possible to prove the conformal invariance directly.

## 7. Concluding remarks

To obtain exact solutions in string theory, it is rather hopeless to start with the field equations expressed as a power series in  $\alpha'$ , and try to solve them explicitly. One must first make an ansatz which simplifies the form of these equations. We have studied such an ansatz, the chiral null models (2.6), and shown that they have the property that there exists a scheme in which the leading order string solutions are exact. This generalizes a number of previous results. The chiral null models include the plane wave type solutions and the fundamental string background which were previously shown to be exact. But as we have seen, they also include, e.g., the solution describing traveling waves along the fundamental string, and, after a dimensional reduction, the extremal electrically charged dilaton black holes and the dilaton IWP solutions. Moreover, there are interesting solutions describing magnetic field configurations. It is rather surprising that such a large class of leading-order solutions turn out to be exact in bosonic, superstring *and* heterotic string theories.

One can, in fact, turn the argument around. Even the leading order string equations (analogous to Einstein's equations) can be rather complicated when the dilaton and antisymmetric tensor are nontrivial. By choosing an ansatz at the level of the string world sheet action which yields simple equations for the  $\sigma$ -model  $\beta$ -functions, one can easily find new solutions of even the leading order equations. The chiral null models are an example of this. Some of the solutions we have discussed, e.g. (4.6) with a general  $K$ , appear to be new.

However, it is clear that not all of the solutions of the leading order equations can be obtained from chiral null models. The chiral coupling, which is an important feature of these models, leads to a no-force condition on the solutions, and the possibility of linear superposition. This happens only for a certain charge to mass ratio which typically characterizes extreme black holes or black strings. Furthermore, we have obtained only four dimensional black-hole type solutions with *electric* charge. Extreme magnetically charged black holes do not appear to be described by chiral null models.

We have considered examples of chiral null models with a flat transverse space. As we have remarked, it is straightforward to extend this class of models to any transverse space which is itself an exact conformal field theory. It may be interesting to explore the new solutions (with non-trivial mixing of “space-time” and “internal” directions) which can be obtained in this way.

An important open problem is to study string propagation in the backgrounds discussed here. This will improve our understanding of the physical properties of these solutions in string theory.

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## Appendix A. Geometrical quantities for the chiral null model

### A.1. Generalized connection

The classical string equations for a  $\sigma$ -model

$$L = C_{\mu\nu}(x)\partial x^\mu \bar{\partial} x^\nu, \quad C_{\mu\nu} \equiv G_{\mu\nu} + B_{\mu\nu}, \quad (\text{A.1})$$

are naturally expressed in terms of the generalized connection with torsion

$$\partial \bar{\partial} x^\lambda + \hat{\Gamma}_{-\mu\nu}^\lambda(x) \partial x^\mu \bar{\partial} x^\nu = 0 \quad \text{or} \quad \partial \bar{\partial} x^\lambda + \hat{\Gamma}_{+\mu\nu}^\lambda(x) \partial x^\nu \bar{\partial} x^\mu = 0, \quad (\text{A.2})$$

$$\hat{\Gamma}_{\pm\mu\nu}^\lambda = \Gamma_{\mu\nu}^\lambda \pm \frac{1}{2} H^\lambda{}_{\mu\nu}, \quad \hat{\Gamma}_{-\mu\nu}^\lambda = \hat{\Gamma}_{+\nu\mu}^\lambda = \frac{1}{2} G^{\lambda\rho} (\partial_\mu C_{\rho\nu} + \partial_\nu C_{\mu\rho} - \partial_\rho C_{\mu\nu}). \quad (\text{A.3})$$

In the case of our model (2.7)  $x^\mu = (u, v, x^i)$  and

$$G_{uv} = \frac{1}{2}F, \quad G_{ui} = FA_i, \quad G_{uu} = FK, \quad G_{vi} = 0, \quad G_{vv} = 0, \quad G_{ij} = \delta_{ij}, \quad (\text{A.4})$$

$$\begin{aligned} G^{uv} &= 2F^{-1}, \quad G^{ui} = G^{uu} = 0, \quad G^{vi} = -2A^i, \\ G^{vv} &= 4(A_i A^i - F^{-1}K), \quad G^{ij} = \delta^{ij}, \\ C_{uv} &= F, \quad C_{vu} = 0, \quad C_{ui} = 2FA_i, \quad C_{iu} = 0, \quad C_{uu} = FK, \\ C_{vi} &= C_{iv} = 0, \quad C_{vv} = 0, \quad C_{ij} = \delta_{ij}. \end{aligned} \quad (\text{A.5})$$

We shall use the following definitions

$$h(x) = \frac{1}{2} \ln F(x), \quad \mathcal{F}_{ij} = \partial_i A_j - \partial_j A_i, \quad K = K(x, u), \quad A_i = A_i(x, u). \quad (\text{A.6})$$

The corresponding components of the connection are ( $\hat{\Gamma}_{+\mu\nu}^\lambda = \hat{\Gamma}_{-\nu\mu}^\lambda$ )

$$\begin{aligned} \hat{\Gamma}_{-ui}^u &= 2\partial_i h, \quad \hat{\Gamma}_{-ij}^u = \hat{\Gamma}_{-vj}^u = \hat{\Gamma}_{-mu}^u = \hat{\Gamma}_{-mv}^u = 0, \\ \hat{\Gamma}_{-uv}^i &= -F\partial^i h, \quad \hat{\Gamma}_{-uu}^i = F\partial_u A^i - \frac{1}{2}\partial^i(FK), \\ \hat{\Gamma}_{-uj}^i &= -A_j\partial^i F - F\mathcal{F}^i{}_j, \quad \hat{\Gamma}_{-v\mu}^i = \hat{\Gamma}_{-j\mu}^i = 0, \\ \hat{\Gamma}_{-iv}^v &= 2\partial_i h, \quad \hat{\Gamma}_{-ij}^v = 2\partial_i A_j + 4A_j\partial_i h, \quad \hat{\Gamma}_{-iu}^v = \partial_i K + 2K\partial_i h, \\ \hat{\Gamma}_{-ui}^v &= \partial_i K - 2K\partial_i h + 2A_i A^j \partial_j F - 2FA^j \mathcal{F}_{ij}, \quad \hat{\Gamma}_{-uv}^v = A^i \partial_i F, \\ \hat{\Gamma}_{-uu}^v &= \partial_u K - 2FA^i \partial_u A_i + A^i \partial_i(FK), \quad \hat{\Gamma}_{-v\mu}^v = 0. \end{aligned} \quad (\text{A.7})$$

It is straightforward to compute the curvature tensor corresponding to  $\hat{\Gamma}_{\pm\mu\nu}^\lambda$  (note that the torsion here is a closed form)

$$\hat{R}_{\pm\mu\nu\rho}^\lambda = \hat{R}_{\mu\nu\rho}^\lambda(\hat{\Gamma}_{\pm\mu\nu}^\lambda), \quad \hat{R}_{-\lambda\mu\nu\rho} = \hat{R}_{+\nu\rho\lambda\mu}. \quad (\text{A.8})$$

We get

$$\begin{aligned} \hat{R}_{-\mu\nu\rho}^u &= 0, \quad \hat{R}_{-j\nu\rho}^i = 0, \quad \hat{R}_{-\mu uv}^\lambda = 0, \quad \hat{R}_{-v\nu\rho}^\mu = 0, \\ \hat{R}_{-ivj}^v &= 2F^{-1}\hat{R}_{-ujv}^i = -2\partial_i\partial_j h, \quad \hat{R}_{-uvj}^v = -2FA^i\partial_i\partial_j h, \\ \hat{R}_{-iuj}^v &= 2F^{-1}\hat{R}_{-uj u}^i = 2\partial_i\partial_u A_j + 4\partial_i h\partial_u A_j - 2K\partial_i\partial_j h - \partial_i\partial_j K - 2\partial_i h\partial_j K, \\ \hat{R}_{-ijk}^v &= -2F^{-1}\hat{R}_{-ujk}^i = 2\partial_i\mathcal{F}_{jk} + 4\partial_i h\mathcal{F}_{jk} + 4A_k\partial_j\partial_i h - 4A_j\partial_k\partial_i h. \end{aligned} \quad (\text{A.9})$$

Note that product of the curvatures  $\hat{R}_{-\lambda\rho}^{mn}\hat{R}_{-mn\mu\nu}$  vanishes.

### A.2. Special holonomy

The expressions for the curvature (A.9) reflect holonomy properties of the generalized connections  $\hat{\Gamma}_{\pm\mu\nu}^\lambda$ . It turns out that the holonomy group of  $\hat{\Gamma}_{-\mu\nu}^\lambda$  is an abelian  $(D-2)$ -dimensional “null” subgroup of the Lorentz group  $SO(1, D-1)$ . The holonomy group of  $\hat{\Gamma}_{+\mu\nu}^\lambda$  is not special for generic functions  $F, K, A_i$ . It becomes the Euclidean group in  $D-2$  dimensions when  $F=1$  and reduces further to its rotational subgroup  $SO(D-2)$  when  $F=K=1, A_i=A_i(x)$ .

It is easy to argue that a special holonomy of the generalized connection  $\hat{\Gamma}_{-\mu\nu}^\lambda$  in (A.2) is a direct consequence of the presence of a chiral current in the  $\sigma$ -model (A.1) (for a related more general discussion see [53] and refs. there).

If one introduces the vierbeins and defines the following differentials (or ‘currents’)

$$\theta^m = e_\mu^m \partial x^\mu, \quad \bar{\theta}^m = e_\mu^m \bar{\partial} x^\mu, \quad G_{\mu\nu} = \eta_{mn} e_\mu^m e_\nu^n, \quad (\text{A.10})$$

where  $\eta_{mn}$  is the tangent space metric, then the string equation (A.2) can be written in the form

$$\bar{\partial}\theta^m + \hat{\omega}_{-n\mu}^m \bar{\partial}x^\mu \theta^n = 0 \quad \text{or} \quad \partial\bar{\theta}^m + \hat{\omega}_{+n\mu}^m \partial x^\mu \bar{\theta}^n = 0, \quad (\text{A.11})$$

where  $\hat{\omega}_{\pm n\mu}^m$  are the generalized Lorentz connections

$$\hat{\omega}_{\pm n\mu}^m = e_\lambda^m \hat{\Gamma}_{\pm\mu\nu}^\lambda e_n^\nu + e_\lambda^m \partial_\nu e_n^\lambda. \quad (\text{A.12})$$

In the case of (2.7) one may choose (the tangent space indices take the following values:  $m = (\hat{u}, \hat{v}, \hat{i})$ )

$$\theta^{\hat{u}} = F \partial u, \quad \theta^{\hat{v}} = \partial v + K \partial u + 2A_i \partial x^i, \quad \theta^{\hat{i}} = \partial x^i, \quad (\text{A.13})$$

so the Lagrangian (2.7) takes the form

$$L = \theta^{\hat{u}} \bar{\theta}^{\hat{v}} + \theta^{\hat{i}} \bar{\theta}^{\hat{j}} + \alpha' \mathcal{R} \phi(x). \quad (\text{A.14})$$

Then the existence of the null  $v$ -isometry implying  $\bar{\partial}\theta^{\hat{u}} = 0$  tells us that  $\hat{\omega}_{-n\mu}^{\hat{u}} = 0$ , i.e. that the connection  $\hat{\omega}_-$  has a reduced holonomy.<sup>25</sup>

Defining the connection 1-forms ( $\eta_{\hat{u}\hat{v}} = \frac{1}{2}$ ,  $\eta_{\hat{i}\hat{j}} = \delta_{\hat{i}\hat{j}}$ )

$$\hat{\omega}_{\pm mn} = \eta_{mp} \hat{\omega}_{\pm n\mu}^p dx^\mu = -\hat{\omega}_{\pm nm}, \quad (\text{A.15})$$

we find  $\hat{\omega}_{-m\hat{v}} = \hat{\omega}_{-\hat{i}\hat{j}} = 0$ , and

$$\hat{\omega}_{-\hat{u}\hat{i}} = \partial_i h dv + \left(\frac{1}{2} \partial_i K - \partial_u A_i + K \partial_i h\right) du + (\mathcal{F}_{ij} + 2A_j \partial_i h) dx^j, \quad (\text{A.16})$$

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<sup>25</sup> In the case of the  $F$ -model [1] one has two null chiral currents ( $u$  and  $v$  are on an equal footing) and so both ‘left’ and ‘right’ connections should have the same properties. Note, however, that our choice of vierbein in (A.13) is not symmetric in  $u$  and  $v$  so an extra Lorentz transformation will be needed to relate  $\hat{\omega}_-$  to  $\hat{\omega}_+$ .

$$\hat{\omega}_{+\hat{u}\hat{v}} = \partial_i h dx^i, \quad \hat{\omega}_{+\hat{i}\hat{j}} = -F \mathcal{F}_{ij} du, \quad (\text{A.17})$$

$$\hat{\omega}_{+\hat{u}\hat{i}} = \left(\frac{1}{2} \partial_i K - \partial_u A_i\right) du, \quad \hat{\omega}_{+\hat{v}\hat{i}} = F \partial_i h du.$$

Since the algebra of the Lorentz group  $SO(1, D-1)$  is generated by  $M \equiv M_{\hat{u}\hat{v}}$ ,  $L_{\hat{i}} \equiv M_{\hat{u}\hat{i}}$ ,  $R_{\hat{i}} \equiv M_{\hat{v}\hat{i}}$ ,  $M_{\hat{i}\hat{j}}$  satisfying, in particular,

$$[M, M_{\hat{i}\hat{j}}] = 0, \quad [M, L_{\hat{i}}] = L_{\hat{i}}, \quad [M, R_{\hat{i}}] = -R_{\hat{i}}, \quad [L_{\hat{i}}, R_{\hat{j}}] = 2\delta_{\hat{i}\hat{j}} M + M_{\hat{i}\hat{j}}, \quad (\text{A.18})$$

$$[M_{\hat{i}\hat{j}}, L_{\hat{k}}] = 4L_{[\hat{j}} \eta_{\hat{i}]\hat{k}}, \quad [M_{\hat{i}\hat{j}}, R_{\hat{k}}] = 4R_{[\hat{j}} \eta_{\hat{i}]\hat{k}}, \quad [L_{\hat{i}}, L_{\hat{j}}] = [R_{\hat{i}}, R_{\hat{j}}] = 0,$$

we conclude that the holonomy group of  $\hat{\omega}_-$  is equivalent to the non-compact abelian subgroup of the Lorentz group generated by  $M_{\hat{u}\hat{i}}$  (it is “null”, having zero norm associated with it). The holonomy of  $\hat{\omega}_+$  is not special in general.<sup>26</sup>

Let us now consider some particular cases. When  $F = \text{const}$  we find that  $\hat{\omega}_{+\hat{u}\hat{v}} = \hat{\omega}_{+\hat{v}\hat{i}} = 0$  and thus the holonomy algebra of  $\hat{\omega}_+$  reduces the Euclidean algebra generated by  $M_{\hat{i}\hat{j}}$  and  $M_{\hat{u}\hat{i}}$ . It reduces further to the algebra of  $SO(D-2)$  when  $K=1$ ,  $A_i = A_i(x)$  (i.e. in the case of the model (4.10)).

In the case of the generalized FS solution related to the black hole type solutions (4.7) we have  $K = F^{-1}$ ,  $A_i = A_i(x)$  in (4.4) so that the non-vanishing components of the connections are

$$\hat{\omega}_{-\hat{u}\hat{i}} = \partial_i h dv + (\mathcal{F}_{ij} + 2A_j \partial_i h) dx^j, \quad (\text{A.19})$$

$$\hat{\omega}_{+\hat{u}\hat{v}} = \partial_i h dx^i, \quad \hat{\omega}_{+\hat{i}\hat{j}} = -F \mathcal{F}_{ij} du,$$

$$\hat{\omega}_{+\hat{u}\hat{i}} = -F^{-1} \partial_i h du, \quad \hat{\omega}_{+\hat{v}\hat{i}} = F \partial_i h du.$$

When  $A_i = 0$  the holonomy algebra of  $\hat{\omega}_+$  becomes the  $2D-3$  dimensional non-semisimple subalgebra of the Lorentz algebra generated by  $M, L_{\hat{i}}$  and  $R_{\hat{i}}$ .

A special holonomy is known [50,51,48] to be related to the presence of extended world sheet supersymmetry in the supersymmetric extensions of the  $\sigma$ -models. In fact, some of the models (2.7) (which, in particular, admit a complex structure) have extended supersymmetry (see Section 6). Let us note also that special holonomy does not guarantee, by itself, conformal invariance since for that the dilaton is crucial as well. Still, it is related (in a proper renormalization scheme) to the on-shell finiteness of our models on a flat world sheet.

### A.3. Parallelizable spaces and connection to WZW models based on non-semisimple groups

One may be interested which of our spaces are parallelizable with respect to the generalized connection, i.e. have  $\hat{R}_{-\mu\nu\rho}^\lambda = 0$  (and thus  $\hat{R}_{+\mu\nu\rho}^\lambda = 0$ , see (A.8)). One expects parallelizable spaces to be related to group spaces and indeed this is what we find.

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<sup>26</sup> In the absence of torsion the irreducible holonomy groups (or “special geometries”) on non-symmetric spaces have been classified [57]. No systematic classification seems to be known in the torsionful case. We thank J. De Boer and G. Papadopoulos for helpful comments on this subject.

Since the string naturally ‘feels’ the generalized connection with torsion, the vanishing of the generalized curvature is the analogue of the flatness condition in the point-particle theory. In particular,  $\hat{R}_\pm = 0$  means that locally  $\hat{\Gamma}_{\pm\mu\nu}^\lambda = f_{\pm n}^{-1\lambda} \partial_\nu f_{\pm\mu}^n$ . Then (A.2) implies the existence of  $D$  chiral and  $D$  antichiral conserved currents  $f_{-\mu}^n(x) \partial x^\mu$  and  $f_{+\mu}^n(x) \bar{\partial} x^\mu$ .

As follows from (A.9), a necessary condition for parallelizability is  $\partial_i \partial_j h = 0$ , i.e.  $h = h_0 + p_i x^i$ . Then the two remaining conditions take the form

$$\hat{R}_{-ijk}^v = \partial_i \mathcal{F}_{jk} + 2p_i \mathcal{F}_{jk} = 0 , \quad (\text{A.20})$$

$$\hat{R}_{-iuj}^v = 2\partial_i \partial_u A_j + 4p_i \partial_u A_j - \partial_i \partial_j K - 2p_i \partial_j K = 0 .$$

In view of the gauge freedom (2.8) we may set  $K = 0$ . If  $p_i \neq 0$  the solution is  $A_i = C_i(u) \exp(-2p_j x^j)$ . By redefining the coordinates  $v' = v + \exp(-2p_i x^i) g(u)$ ,  $x'^i = x^i + w^i(u)$  the corresponding model can be transformed into the product of the  $SL(2, R)$  WZW model (cf. (B.7)) and  $R^{D-3}$ .

The case of  $p_i = 0$ , i.e.  $F = \text{const}$  is more subtle. The solution is  $A_i = C_i(u) - \frac{1}{2} \mathcal{F}_{ij} x^j$ ,  $\mathcal{F}_{ij} = \text{const}$ . One can further eliminate  $C_i$  by a coordinate transformation  $v' = v + q(u) + s_i(u) x^i$ ,  $x'^i = x^i + w^i(u)$ . We are finally left with the following model (cf. (4.10))

$$F = 1 , \quad K = 0 , \quad A_i = -\frac{1}{2} \mathcal{F}_{ij} x^j . \quad (\text{A.21})$$

These spaces can be interpreted as boosted products of group spaces, or, equivalently, as spaces corresponding to WZW models for non-semisimple groups. To show this one should first put  $\mathcal{F}_{ij}$  into the block-diagonal form by a coordinate  $x^i$  rotation, so that its elements are represented by constants  $\mathcal{H}_1, \dots, \mathcal{H}_{[D/2-1]}$  and the corresponding Lagrangian (4.10) is (we split  $x^i$  into pairs representing 2-planes;  $a, b = 1, 2$ )

$$L = \partial u \bar{\partial} v + \sum_{s=1}^{[D/2-1]} (\mathcal{H}_s \epsilon_{ab} x_s^a \partial u \bar{\partial} x_s^b + \partial x_s^a \bar{\partial} x_{as}) . \quad (\text{A.22})$$

The first non-trivial case is that of  $D = 4$ , i.e.  $\mathcal{F}_{ab} = \mathcal{H} \epsilon_{ab}$ . The corresponding model ( $x_1 = r \cos \theta$ ,  $x_2 = r \sin \theta$ )

$$L = \partial u \bar{\partial} v + \mathcal{H} \epsilon_{ab} x^a \bar{\partial} x^b \partial u + \partial x^a \bar{\partial} x_a \quad (\text{A.23})$$

$$= \partial u \bar{\partial} v + \mathcal{H} r^2 \bar{\partial} \theta \partial u + \partial r \bar{\partial} r + r^2 \partial \theta \bar{\partial} \theta ,$$

is equivalent to the  $E_2^c$  WZW model of ref. [19] (note that  $\mathcal{H}$  can be set equal to  $-1$  by a rescaling of  $u, v$ ). In fact, the coordinate transformation [58]  $x_1 = y_1 + y_2 \cos u$ ,  $x_2 = y_2 \sin u$ ,  $v = v' + y_1 y_2 \sin u$  puts (A.23) in the form

$$L = \partial u \bar{\partial} v' + \partial y_1 \bar{\partial} y_1 + \partial y_2 \bar{\partial} y_2 + 2 \cos u \partial y_1 \bar{\partial} y_2 , \quad (\text{A.24})$$

which is obtained from the  $R \times SU(2)$  WZW action by a singular boost and rescaling of the level  $k$  or  $\alpha'$  (see [59]). If  $s$  is a time-like coordinate of the  $R$ -factor and  $\psi$  is an angle of  $SU(2)$  one should set  $s = u$ ,  $\psi = u + \epsilon v$ , rescale  $k$  and  $y_i$  by powers of  $\epsilon$  and

take  $\epsilon$  to zero. The  $D = 5$  model (A.23) is equivalent to the product of the  $D = 4$  model and a free space-like direction. The  $D = 6$  model (which contains two sets of planar coordinates  $x_s^a$ ,  $s = 1, 2$ ) is equivalent to the non-semisimple or boosted version of the  $SL(2, R)_{-k_1} \times SU(2)_{k_2}$  WZW model (see eq. (4.16) in [59]). The required coordinate transformation is  $\psi_1 = u, \psi_2 = u + \epsilon v$ , etc. The non-trivial parameter  $\mathcal{H}_1/\mathcal{H}_2$  is equal to the ratio of the levels  $k_1/k_2$ .

The next non-trivial model is with  $D = 8$ . It can be obtained by boosting  $SL(2, R)_{-k_1} \times SU(2)_{k_2} \times SU(2)_{k_3}$  WZW model ( $\psi_1 = u, \psi_2 = u + \epsilon v - \epsilon \lambda, \psi_3 = u + \epsilon \lambda$ ) with the direction  $\lambda$  decoupling in the limit  $\epsilon \rightarrow 0$ . All higher  $D$  models are related to similar WZW models based on direct products of  $SL(2, R)_k$ ,  $SU(2)_k$  and  $R$  factors, or, equivalently, on corresponding non-semisimple groups. The parameters  $\mathcal{H}_s$  are essentially equivalent to the rescaled levels  $k_n$  of the factors.

Finally, it is interesting to note that all the models (A.21), like the  $D = 4$  model (A.23), can be related to the flat space model in the same way as this was shown [60, 58] for the  $D = 4$  model of [19]. In fact, let us consider one pair of planar coordinates  $x^a$  and gauge the rotational symmetry in the plane. We get the following model

$$L = \partial u \bar{\partial} v + \partial r \bar{\partial} r + r^2(\partial \theta + A)(\bar{\partial} \theta + \bar{A}) + \mathcal{H} r^2 \partial u (\bar{\partial} \theta + \bar{A}) + \bar{A} \partial \tilde{\theta} - A \bar{\partial} \tilde{\theta} , \quad (\text{A.25})$$

where  $\tilde{\theta}$  is the dual coordinate and  $A, \bar{A}$  are components of the 2d gauge field. Shifting  $A$  by  $-\mathcal{H} \partial u$  and  $v$  by  $\mathcal{H} \tilde{\theta}$  we get a model which is equivalent to the flat space one. The same transformation can be done independently for each plane. The original  $\sigma$ -model (A.21) is thus related to a flat space one by a combination of duality, coordinate transformation and “inverse” duality. If, however, the true starting point is the “doubled” or “gauged” model (A.25), then the transformation to the model corresponding to the flat space is just a coordinate transformation on the extended configuration space of  $(u, v, r, \theta, \tilde{\theta}, A, \bar{A})$ .

#### A.4. Leading-order conformal invariance equations

The standard leading-order conformal invariance conditions are

$$\hat{R}_{-\mu\nu} + 2\hat{D}_{-\mu}\hat{D}_{-\nu}\phi = 0 , \quad (\text{A.26})$$

where  $\hat{R}_{\pm\mu\nu} = \hat{R}_{\mp\nu\mu} = \hat{R}_{\pm\mu\lambda\nu}^\lambda$  and  $\hat{D}_{\pm\mu}$  are the Ricci tensor and covariant derivative for the connection  $\hat{\Gamma}_{\pm\mu\nu}^\lambda$  (the symmetric and antisymmetric parts of (A.26) give equations for  $G_{\mu\nu}$  and  $B_{\mu\nu}$ ). Computing the Ricci tensor from (A.9) one finds

$$\hat{R}_{-uv} = -F\partial^i\partial_i h , \quad \hat{R}_{-ij} = -2\partial_i\partial_j h , \quad \hat{R}_{-iu} = \hat{R}_{-iv} = \hat{R}_{-v\mu} = 0 , \quad (\text{A.27})$$

$$\hat{R}_{-uu} = -F\left(\frac{1}{2}\partial^i\partial_i K + \partial^i h \partial_i K + K \partial^i\partial_i h - \partial^i\partial_u A_i - 2\partial^i h \partial_u A_i\right) ,$$

$$\hat{R}_{-ui} = -F(\partial_j \mathcal{F}^j_i + 2\partial_j h \mathcal{F}^j_i + 2A_i \partial^j \partial_j h) .$$

Then (A.26) implies

$$-\partial_i \partial_j h + \partial_i \partial_j \phi = 0 , \quad \partial_i \partial_u \phi = 0 , \quad \phi(x, u) = \phi(u) + b_i x^i + h(x) , \quad (\text{A.28})$$

and finally we get the same relations as in (5.16), (5.17)

$$-\frac{1}{2}\partial^2 F^{-1} + b^i \partial_i F^{-1} = 0 , \quad -\frac{1}{2}\partial_i \mathcal{F}^{ij} + b_i \mathcal{F}^{ij} = 0 , \quad (\text{A.29})$$

$$-\frac{1}{2}\partial^2 K + b^i \partial_i K + \partial^i \partial_u A_i - 2b^i \partial_u A_i + 2F^{-1}\partial_u^2 \phi = 0 . \quad (\text{A.30})$$



## Appendix B. General $D = 3$ chiral null model

As was shown in [1] the generic  $D = 3$   $F$ -model (2.3) is equivalent to a special  $[SL(2, R) \times R]/R$  gauged WZW model and can also be identified with the extremal limit of the charged black string solution of [61]. Here we shall consider the generic  $D = 3$  model belonging to the class (2.7),

$$L_3 = F(x)\partial u [\bar{\partial}v + K(x, u)\bar{\partial}u + 2A(x, u)\bar{\partial}x] + \partial x\bar{\partial}x + \alpha'\mathcal{R}\phi(x, u) . \quad (\text{B.1})$$

Since the transverse space here is one-dimensional, one can set  $A = 0$  by a transformation of  $v$  (see (2.8)). The functions  $F, K$  and  $\phi$  are then subject to (see (2.10)–(2.12))

$$\partial_x^2 F^{-1} = 2b\partial_x F^{-1} , \quad \partial_x^2 K = 2b\partial_x K - 4F^{-1}\partial_u^2 \phi , \quad \phi = \phi(u) + bx + \frac{1}{2}\ln F(x) . \quad (\text{B.2})$$

Assuming for simplicity that  $K$  and  $\phi$  do not depend on  $u$  we get the following solutions

$$F^{-1} = a + me^{2bx} , \quad K = a' + m'e^{2bx} = c + nF^{-1}(x) , \quad (\text{B.3})$$

so that by shifting  $v$  we finish with the following conformal  $D = 3$  model

$$L_3 = F(x)\partial u\bar{\partial}v + n\partial u\bar{\partial}u + \partial x\bar{\partial}x + \alpha'\mathcal{R}\phi(x) , \quad (\text{B.4})$$

$$F^{-1} = a + me^{2bx} , \quad \phi(x) = \phi_0 - \frac{1}{2}\ln(ae^{-2bx} + m) . \quad (\text{B.5})$$

$a, n, m$  are arbitrary constants which take only two non-trivial values: 0 and 1 (–1 case is related to the +1 one by an analytic continuation). In what follows we shall set  $m = 1$ . The  $n = 0$  model is the  $F$ -model discussed in [1]. In what follows we shall keep  $n$  general thus treating both  $n = 0$  and  $n = 1$  cases at the same time.

The solution (B.3) with  $a = 0$  has a constant dilaton and thus the corresponding model must be equivalent to the  $SL(2, R)$  WZW model (since there are no other  $\phi = \text{const}$  solutions in  $D = 3$  in a properly chosen scheme [1]). In fact, the  $a = 0$  model

$$L_3 = e^{-2bx}\partial u\bar{\partial}v + n\partial u\bar{\partial}u + \partial x\bar{\partial}x + \alpha'\mathcal{R}\phi_0 , \quad (\text{B.6})$$

is related to the  $SL(2, R)$  WZW Lagrangian written in the Gauss decomposition parametrization (we follow the notation of [1] and set  $\alpha' = 1$ )

$$L_{wzw} = k (e^{-2r}\partial u\bar{\partial}v + \partial r\bar{\partial}r) , \quad (\text{B.7})$$

by the following coordinate transformation ( $u', v'$  stand for the coordinates in (B.7))

$$u' = \frac{1}{2\sqrt{n}}e^{2b\sqrt{n}u} , \quad v' = bv - \sqrt{n}e^{2bx} , \quad r = bx + b\sqrt{n}u , \quad b^2 = 1/k . \quad (\text{B.8})$$

### B.1. Gauged WZW model interpretation

Like the  $n = 0$  model, the  $n = 1$  one (B.6) can be related to a special  $[SL(2, R) \times R]/R$  gauged WZW model. This provides an explicit illustration of our claim that the chiral

null backgrounds are exact conformal models.<sup>27</sup> The  $SL(2, R) \times R$  WZW model written in the Gauss decomposition parametrization, i.e. (B.7) with an additional  $R$ -term  $\partial y \bar{\partial} y$ , has the following obvious global symmetries: independent shifts of  $u, v, y$  and shifts of  $r$  combined with rescalings of  $u$  and (or)  $v$ . Gauging the translational subgroup

$$u \rightarrow u + \epsilon, \quad v \rightarrow v + \epsilon, \quad y \rightarrow y + \rho\epsilon, \quad \rho = \text{const}, \quad (\text{B.9})$$

fixing the gauge  $y = 0$  and integrating out the two dimensional gauge field, one gets the  $n = 0$  model (B.4) with  $a = \rho^{-2}$  [1]. The subgroup which is to be gauged to get the  $n = 1$  model is<sup>28</sup>

$$u \rightarrow e^{2\sqrt{n}\epsilon} u, \quad v \rightarrow v + \epsilon, \quad r \rightarrow r + \sqrt{n}\epsilon, \quad y \rightarrow y + \rho\epsilon. \quad (\text{B.10})$$

In view of (B.8) this is just the translational symmetry (B.9) (with  $\epsilon \rightarrow b^{-1}\epsilon$ ) of the action (B.6) (with  $\partial y \bar{\partial} y$  added). Since (B.6) is a coordinate transformation of the WZW action (B.7) we can start directly with (B.6) in the gauging procedure,

$$L_{gwzw} = e^{-2bx} (\partial u + A)(\bar{\partial} v + \bar{A}) + n(\partial u + A)(\bar{\partial} u + \bar{A}) + \partial x \bar{\partial} x + (\partial y + \rho A)(\bar{\partial} y + \rho \bar{A}). \quad (\text{B.11})$$

Fixing  $y = 0$  as a gauge and integrating out  $A, \bar{A}$  we get

$$\begin{aligned} L_{gwzw} = & \frac{\rho^2 e^{-2bx}}{\rho^2 + n + e^{-2bx}} \partial u \bar{\partial} v + \frac{n \rho^2}{\rho^2 + n + e^{-2bx}} \partial u \bar{\partial} u \\ & + \partial x \bar{\partial} x + \alpha' \mathcal{R}[\phi'_0 + \frac{1}{2} \ln(\rho^2 + n + e^{-2bx})]. \end{aligned} \quad (\text{B.12})$$

The redefinition

$$u' = (1 + n\rho^{-2})^{1/2} u, \quad v' = (1 + n\rho^{-2})^{1/2} (v + \frac{n}{\rho^2 + n} u), \quad (\text{B.13})$$

puts this action into the desired form (B.4), (B.5) with  $a = (\rho^2 + n)^{-1}$ .

## B.2. Extremal black string interpretation

The generic  $D = 3$   $F$ -model (i.e. (B.4) with  $n = 0$ ) can be considered as an extremal limit of the charged black string solution of [61]. Here we point out that a similar statement is true for the  $n = 1$  model (B.4). This is a particular case of the relation between the model (2.21) and the charged black string solution discussed in Section 2.4 (see (2.22)). Starting with the non-extremal charged black string  $\sigma$ -model which has the metric

$$ds^2 = -f_1(r') dt'^2 + f_2(r') dy'^2 + h(r') dr'^2, \quad (\text{B.14})$$

---

<sup>27</sup> It is very likely that there exists a generalization of the nilpotent gauging procedure of ref. [5] which makes it possible to identify not just one  $D = 3$  model but a whole subclass of the chiral null models with  $F^{-1} = \sum_{i=1}^d e^{\alpha_i \cdot x}$  ( $\alpha_i$  are the simple roots of the algebra of a maximally non-compact Lie group of rank  $d = D - 2$ ) with the gauged WZW models.

<sup>28</sup> It may be useful to recall that the subgroup that leads to the charged black string of [61] is [1]:  $u \rightarrow e^\epsilon u, \quad v \rightarrow e^\epsilon v, \quad r \rightarrow r + \epsilon, \quad y \rightarrow y + \rho\epsilon$ .

$$f_1 = (1 - \frac{M_1}{r'}) , \quad f_2 = (1 - \frac{M_2}{r'}) , \quad h = (4r'^2 f_1 f_2)^{-1} , \quad M_1 = M , \quad M_2 = \frac{Q^2}{M} , \quad (\text{B.15})$$

boosting the solution

$$t = \lambda v + (\frac{1}{2}\lambda - \lambda^{-1})u , \quad y = \lambda^{-1}u , \quad \lambda \equiv (\frac{M_1}{M_2} - 1)^{1/2} , \quad (\text{B.16})$$

and then taking the extremal limit  $M \rightarrow Q$ , i.e.  $M_1 \rightarrow M_2$  or  $\lambda \rightarrow 0$  in the resulting  $\sigma$ -model one finishes with the model (B.4) with the metric

$$ds^2 = 2(1 - \frac{M}{r'})dudv + du^2 + h(r')dr'^2 = F(x)dudv + du^2 + dx^2 . \quad (\text{B.17})$$

So the generic  $u$ -independent  $D = 3$  chiral null model can be obtained as an extremal limit of a black string solution.

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